IMPARTIAL COMBINATORIAL MISÈRE GAMES

by
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Abstract

Combinatorial games are played under two different play conventions: normal play, where the last player to move wins, and misère play, where the last player to move loses. Although much work has been done on games played under the normal play convention, less has been done for games played under the misère play convention. This thesis discusses the theory of impartial combinatorial games which are played under the misère game convention. We present a full discussion, including proofs, on Conway's Genus Theory and use genus theory to study two impartial combinatorial games played under the misère game convention. We conclude with an overview of Plambeck's development for impartial combinatorial games played under the misère game convention, the Indistinguishablility Quotient.

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Chapter 1

Introduction

In all future sections, unless otherwise referenced, theorems and/or proofs are the author's own work.

In this chapter we introduce the basics of combinatorial game theory, taking and breaking games, and the differences between games played under the normal play convention and games played under the misère play convention.

1.1 Combinatorial Games

Definition. A **combinatorial game** is a game in which the following conditions are satisfied:

- There are two players, usually denoted by Left and Right, who alternate turns. Generally Left is male whereas Right is female.

- There is a clearly defined rule set which determines the legal moves for each player.

- There is complete information. That is, all information regarding the game is available to both Left and Right at every point of the game.

- There is no element of chance which can affect the game, such as dice, cards, or spinners.

- There is only a finite number of moves allowed and the game will end with one winner and one loser.

- The way to determine the winner and the loser depends on the play convention. Under the **misère play convention**, the first player unable to make a valid move wins. Under the **normal play convention**, the first player unable to make a valid move loses. Equivalently, under the misère play convention, the last player to move loses, while under the normal play convention, the last player to move wins.

All games considered in this thesis will be combinatorial games, even if *combinatorial* is not explicitly stated.

Many games with which we are familiar do not satisfy all the conditions required to be a combinatorial game. For example, most card games, such as Poker, would no longer be enjoyable with complete information. Other games, such as some board games like Monopoly, have the potential for an infinite number of moves, as well as using elements of chance, namely dice. Games such as Tic-Tac-Toe, which have two players, clear rules, perfect information, and a finite number of moves, also have the potentials for draws.

An example of a combinatorial game is the game of **Nim**: given several heaps of tokens, on her turn, a player picks a heap and removes some tokens from that heap. Play continues until no heaps remain. Under the normal play convention, the player who takes the last token is the winner. Under the misère play convention, the player who takes the last token is the loser. The game of Nim, although seemingly trivial, turns out to be extremely useful in certain types of combinatorial game analysis.

Nim is a game to which we will refer frequently, and so, we will use the following shorthand:

Notation. We will denote a Nim heap with n tokens by $\mathbb{n}$.

1.1.1 Options and Followers

Definition. Suppose we are given a game position G. A **left option** of G is a new position which arises after one move from Left. Similarly, a **right option** of G is a new position which arises after one move from Right. The **options** of a position is the union of the left options and the right options.

Notation. We let $\mathcal{G}^L$ and $\mathcal{G}^R$ denote the set of Left and Right options of G respectively.

Definition. Suppose we are given a position in the game G. A **follower** of the position is a new position which can be reached from the initial position after a finite number of moves.

Example 1.1.1. Suppose we are playing the following game: given a heap of n tokens, Left can take one or two tokens, whereas Right can take three. Suppose we are given a heap of size 12. Then the left options of this position are heaps of 11 and 10, while the right option of this position is a heap of size nine. A heap of size three is a follower of a heap of size 12 under the following move set:

$$12 \xrightarrow{L} 11 \xrightarrow{R} 8 \xrightarrow{L} 6 \xrightarrow{R} 3.$$

1.1.2 Games Defined In Terms of Their Options

We define positions based on their options. For a position G, we think of G as a specific position in the game along with its sets $\mathcal{G}^L$ and $\mathcal{G}^R$, and we write

$$G = \{\mathcal{G}^L \mid \mathcal{G}^R\}.$$

Thus, we can define games recursively based on their left and right options from any given position.

Example 1.1.2. Suppose we are playing Nim. Then

$$\mathbb{4} = \{\mathbb{3}, \mathbb{2}, \mathbb{1}, \mathbb{0} \mid \mathbb{3}, \mathbb{2}, \mathbb{1}, \mathbb{0}\}.$$

1.1.3 Disjunctive Sum

Definition. Given combinatorial games $G_1, G_2, \cdots G_\mu$, the **disjunctive sum** is the game $G_1 + G_2 + \cdots G_\mu$ where on a given player's turn, she picks a game G_i and plays in it according to the rules of G_i. In misère play, when a player has no moves in every game in the disjunctive sum, she wins. In normal play, when a player has no moves in every game in the disjunctive sum, she loses.

Notation. Let $\mathcal{X}$ be a set of games, and G a game. Then

$$G + \mathcal{X} = \{G + X \mid X \in \mathcal{X}\}.$$

In the game value notation shown in Section 1.1.2, the disjunctive sum of games G and H is written as:

$$G + H = \{(\mathcal{G}^L + H) \cup (G + \mathcal{H}^L) | (\mathcal{G}^R + H) \cup (G + \mathcal{H}^R)\},$$

where we often remove the $\cup$ and replace it by a comma, i.e.

$$G + H = \{\mathcal{G}^L + H, G + \mathcal{H}^L | \mathcal{G}^R + H, G + \mathcal{H}^R\}.$$

1.1.4 Perfect Play

For this thesis, we will always assume that the players **play perfectly**, that is, both players make the optimal moves available to them. If on her turn, a player can make a move which ensures that she will win, no matter what the other player's subsequent moves are and assuming that she continues to play perfectly, then she will make it. If no such move exists, she will make a non-winning move.

1.1.5 Outcome Classes

Definition. Every combinatorial game G belongs to an **outcome class** which specifies who has the winning strategy. G belongs to one of four disjoint outcome classes as follows:

1) $G \in \mathcal{L}$ if Left has a winning strategy regardless of moving first or second.

2) $G \in \mathcal{R}$ if Right has a winning strategy regardless of moving first or second.

3) $G \in \mathcal{N}$ if the next player to move has a winning strategy.

4) $G \in \mathcal{P}$ if the next player moving does not have a winning strategy (i.e. she will lose). The $\mathcal{P}$ stands for previous, as if the next player loses, the player who would have played previous to her will win, supposing optimal moves.

Theorem 1.1.1. *([4], p.73) Every game G is in exactly one of the four outcome classes.*

For a given position G, the outcome class of G played under the normal play convention may or may not be the same as the outcome class of G played under the misère play convention, as the following two examples show.

Example 1.1.3. Suppose G is a game with no options. Under the normal play convention, the next player to go loses, regardless of whether they are Left or Right, so $G \in \mathcal{P}$. Under the misère play convention, the next player to go wins, regardless of whether they are Left or Right, so $G \in \mathcal{N}$.

Example 1.1.4. Consider the game $\mathbb{2} + \mathbb{3}$. Suppose we are playing under the normal play convention, with Left going first. Consider the following move:

$$\mathbb{2} + \mathbb{3} \xrightarrow{L} \mathbb{2} + \mathbb{2}.$$

Right can respond in one of two ways. The first is

$$\mathbb{2} + \mathbb{2} \xrightarrow{R} \mathbb{2}.$$

However, Left can take the heap of size two,

$$\mathbb{2} \xrightarrow{L} \mathbb{0},$$

leaving Right with no moves.

Right's second response is

$$\mathbb{2} + \mathbb{2} \xrightarrow{R} \mathbb{1} + \mathbb{2}.$$

However, if Left responds with

$$\mathbb{1} + \mathbb{2} \xrightarrow{L} \mathbb{1} + \mathbb{1},$$

Right is forced to respond with

$$\mathbb{1} + \mathbb{1} \xrightarrow{R} \mathbb{1},$$

and Left takes the last token,

$$\mathbb{1} \xrightarrow{L} \mathbb{0}$$

leaving Right with no moves.

Interchanging Right and Left and repeating the argument, we see that $\mathbb{2} + \mathbb{3} \in \mathcal{N}$ under the normal play convention.

Suppose we are playing under the misère play convention, with Left going first. Consider the following move:

$$\mathbb{2} + \mathbb{3} \xrightarrow{L} \mathbb{2} + \mathbb{2}.$$

Right can respond in one of two ways. The first is

$$\mathbb{2} + \mathbb{2} \xrightarrow{R} \mathbb{2},$$

However, Left can take one of the tokens from the heap of size two

$$\mathbb{2} \xrightarrow{L} \mathbb{1},$$

which forces Right to take the heap of size one,

$$\mathbb{1} \xrightarrow{R} \mathbb{0},$$

leaving Left with no moves. Since we are now playing under the misère game convention, Left wins.

Right's second response is

$$\mathbb{2} + \mathbb{2} \xrightarrow{R} \mathbb{1} + \mathbb{2}.$$

However, if Left responds with

$$\mathbb{1} + \mathbb{2} \xrightarrow{L} \mathbb{1},$$

Right is forced to take the heap of size one,

$$\mathbb{1} \xrightarrow{R} \mathbb{0},$$

leaving Left without any moves. Again Left wins.

Interchanging Right and Left and repeating the argument, we see that $\mathbb{2} + \mathbb{3} \in \mathcal{N}$ under the misère play convention as well.

1.1.6 Impartial and Partizan

Based on the moves available to each player, combinatorial games are partitioned as follows:

Definition. A combinatorial game G is **impartial** if from any position and for all of its followers, the left and right options are equal. Otherwise, the game is **partizan**.

Nim is an impartial game since from any position, either player has exactly the same options. However, under a slight variation, where Left can only take one token whereas Right can only take two, this game becomes partizan since given a heap with one token, Left has a move while Right does not.

Theorem 1.1.2. *([1], p.41) If G is an impartial combinatorial game, then the outcome class of G is either $\mathcal{N}$ or $\mathcal{P}$.*

Note that Theorem 1.1.2 is true for impartial games played under the normal or the misère play convention. That is, an impartial game played under the normal play convention is in either $\mathcal{N}$ or $\mathcal{P}$ and an impartial game played under the misère play convention is in either $\mathcal{N}$ or $\mathcal{P}$.

1.1.7 Equivalence and Normal Play Values

Definition. Given two games G, and H played under the normal play convention, we write $G = H$ and say that G **is equivalent to** H if for all games X played under the normal play convention, $G + X$ has the same outcome as $H + X$, where $+$ denotes the disjunctive sum from Section 1.1.3 and we are playing both disjunctive sums under the normal play convention.

Thus, if $G = H$, then in any disjunctive sum of games, we can replace G with H without any effect on the outcome class of the disjunctive sum.

Similar to how games were defined in terms of their options in Section 1.1.2, under the normal play convention, games are also recursively assigned values based on the values of their options. For a game G, we write

$$G = \{\mathcal{G}^L | \mathcal{G}^R\}.$$

The most basic position of a game is a position in which neither Left nor Right has options, i.e. $G^L = G^R = \emptyset$. We then write the game as $\{\ |\ \}$. If we are playing under the normal play convention, the value of this game is 0. We now construct three more games:

$$\{0|0\}, \{0|\ \}, \{\ |0\},$$

where, if we are playing under the normal play convention, we denote

$$\begin{aligned}\{0|0\} &:= *,\\ \{0|\ \} &:= 1,\\ \{\ |0\} &:= -1.\end{aligned}$$

The reason for the nomenclature of 1 and -1 is that these games share similar properties to what we generally think of as 1 and -1 if we are playing under the normal play convention ([4], p.6-12).

We can continue to build games as such, giving us, under the normal play convention, games which are "equal" to any integer or dyadic rational, and "ordered" in the same way as their numerical counterparts, as well as games which are called infinitesimal, which are neither greater than nor less than zero. For more on this, see [4], Chapters Zero, One, Eight, and Nine.

One important thing to note is that two games can have different options, but the same game value. For example, there are other games whose value is 0, even though there are options for both Left and Right.

It seems that we have used = for both equivalence and game value. The reason for this is:

Theorem 1.1.3. *([4], p. 112) Given games G and H. The values G and H are equal $\iff G = H$.*

Equivalence and game values are only for games played under the normal play convention. **Under the misère play convention, we have neither a notion of game values nor game equivalence**. This is because in misère play, a game with no options is considered a winning game for the next player, even though the player has no moves available to her, which causes some difficulties in determining the equivalence between certain games. To quote Conway:

> Unfortunately, the complications so produced persist indefinitely, and make the misère play theory much more complicated than the normal one. ([4], p. 138)

1.2 Taking and Breaking Games

Taking and Breaking games are impartial games involving removing tokens from heaps and/or splitting heaps of tokens into smaller heaps, based on the rules of the game. The name comes from the legal moves - *Taking* from the moves in which one takes tokens, *Breaking* from the moves in which one splits, or breaks up, heaps into smaller heaps.

Definition. An **octal game** is a Taking and Breaking game in which players either take tokens from a heap or take tokens from a heap and then split the remaining tokens into two non-empty heaps. The rules are encoded in a sequence $0.d_1d_2d_3\cdots$, where $d_i \in \{0, 1, 2, \cdots, 7\}$. A legal move is to remove i tokens from a heap and partition the remaining tokens into a, b, or c heaps given that $d_i = 2^a + 2^b + 2^c$.

Example 1.2.1. Consider the octal game 0.12305. We can take $1, 2, 3$ or 5 tokens based on the following rules:

1) Remove one token from a heap provided that the heap's remaining tokens can be partionned into no heaps; that is, provided the initial heap contains exactly one token ($1 = 2^0$).

2) Remove two tokens from a heap provided that the heap's remaining tokens can be partionned into exactly one heap; that is, provided the initial heap contains strictly greater than two tokens ($2 = 2^1$).

3) Remove three tokens from a heap provided that the heap's remaining tokens can be partionned into either no heaps or one non-empty heap; that is, we can always remove three tokens in the heap contains three or more tokens ($3 = 2^0 + 2^1$).

4) We cannot remove four tokens.

5) Remove five tokens from a heap provided that the heap's remaining tokens can be partionned into no heaps or two non-empty heaps; that is, we can removed five tokens from a heap of size five, or we can remove five tokens from a heap of size seven or greater, and then split the remaining tokens from the initial heap into two non-empty heaps ($5 = 2^0 + 2^2$).

6) We cannot remove six or more tokens.

Definition. A **subtraction game** is an octal game where $\forall\ i \in \mathbb{N}$, $d_i = 0$ or 3. That is, the only available moves are to remove i tokens, for i where $d_i = 3$. For a fixed subtraction game, let

$$X = \{i \in \mathbb{N} \mid d_i = 3\}.$$

We call X the **subtraction set** of the game. When referencing a subtraction game, rather than giving the octal code, we sometimes refer only to the subtraction set.

Notation. We let $0.a_1a_2a_3\overline{a_4a_5a_6}$ denote

$$0.\quad a_1a_2a_3\quad a_4a_5a_6\quad a_4a_5a_6\quad a_4a_5a_6\quad a_4a_5a_6\quad a_4a_5a_6\quad a_4a_5a_6\quad \cdots .$$

That is, an overline denotes infinite repetition of the numbers underneath.

Example 1.2.2. The game of Nim is a subtraction game with octal code $0.\bar{3}$. The subtraction set of Nim is $\mathbb{N}$.

Example 1.2.3. The modification of Nim where we can only take prime numbers less than ten is a subtraction game with octal code 0.0330303. Its subtraction set is $\{2, 3, 5, 7\}$.

1.3 Nim and The Sprague-Grundy Theory for Normal Play Impartial Games

Definition. We define **nimbers** recursively as follows:

$$\begin{aligned} 0 &:= \{\,|\,\}, \\ * &:= \{0|0\}, \end{aligned}$$

$$\begin{aligned} *_2 &:= \{0, * | 0, *\}, \\ *_3 &:= \{0, *, *_2 | 0, *, *_2\}, \\ &\cdots \\ *_n &:= \{0, *, *_2, *_3, \cdots, *_{n-1} | 0, *, *_2, *_3, *_{n-1}\}. \end{aligned}$$

Some books denote 0 by $*_0$ and $*$ by $*_1$.

Proposition 1.3.1. *([1], p. 110) A Nim heap with n tokens, $\mathbb{n}$, has game value $*_n$.*

Proof. We proceed by induction on n.

We can see that a Nim heap with no tokens, in which neither Left nor Right have any options, has game value 0 since $\mathbb{0} = \{\ |\ \}$.

Suppose now we have $\mathbb{k}$. From $\mathbb{k}$, we can move to $\mathbb{0}, \mathbb{1}, \cdots, \mathbb{k} - \mathbb{1}$, which have values $0, *, \cdots *_{k-1}$, by induction. Therefore

$$\mathbb{k} = \{0, *, *_2, *_3, \cdots, *_{k-1} | 0, *, *_2, *_3, *_{k-1}\} = *_k,$$

as required. ■

Nim and nimbers are extremely important in impartial normal game analysis because of the following, arrived at independently by both Sprague and Grundy in the 1930s:

Theorem 1.3.2. *([6]) Every normal play impartial game is equivalent to a Nim-heap of a certain size. That is, the game value of an impartial game G is $*_n$ for some $n \in \mathbb{N}$.*

1.3.1 Mex

In impartial normal game calculations, we often make use of the following tool:

Definition. The **minimal excludant**, or **mex**, of a set of ordinals $\mathcal{X}$, is the least ordinal not in the set $\mathcal{X}$.

Example 1.3.1.

$$\begin{aligned} \text{mex}\{1, 2, 3\} &= 0, \\ \text{mex}\{0, 1, 3, 6, 7, 12, 89\} &= 2, \\ \text{mex}\{0, 2, 4, 6, 8, \cdots\} &= 1. \end{aligned}$$

Notation. Let $\mathcal{A} \subseteq \mathbb{Z}^{\geq 0}$. Let $a_1, a_2, \cdots, a_n \in \mathbb{Z}^{\geq 0}$. We abuse notation by letting

$$\text{mex}\{a_1, a_2, \cdots, a_n, \mathcal{A}\} = \text{mex}\{\{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_n\} \cup \mathcal{A}\}.$$

We use mex in the following:

Proposition 1.3.3. *([1], p. 111) Given an impartial game G played under the normal play convention, then G is equivalent to the Nim heap which corresponds to the least possible number that is not the size of any of the heaps which correspond to the options of G. In other words, if the options of G correspond to Nim heaps of size $a, b, c, \cdots$, then G corresponds to a Nim heap of size mex$\{a, b, c, \cdots\}$.*

Combining Proposition 1.3.3 and the Sprague-Grundy Theorem for impartial normal games we obtain the following result:

Proposition 1.3.4. *([1], p.112) Given an impartial game G played under the normal play convention:*

$$\textit{the outcome class of } G = \begin{cases} \mathcal{P} & \textit{if } G \textit{ is equivalent to a Nim heap of size 0} \\ \mathcal{N} & \textit{else.} \end{cases}$$

1.3.2 Nim Sum

Definition. Given non-negative integers n and m, their **Nim sum**, denoted by $n \oplus m$, is the *exclusive or* of their binary representation. Equivalently, the Nim sum of n and m can be determined by writing each of them as a sum of distinct powers of two and then cancelling any power of two which occurs an even number of times.

Example 1.3.2. $11 \oplus 22 \oplus 33$

$$\begin{array}{rcrrrrrr} & & 32 & 16 & 8 & 4 & 2 & 1 \\ \hline 11 & = & & & 1 & 0 & 1 & 1 \\ 22 & = & & 1 & 0 & 1 & 1 & 0 \\ 33 & = & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 60 & = & 1 & 1 & 1 & 1 & 0 & 0 \end{array},$$

or

$$
\begin{aligned}
11 \oplus 22 \oplus 33 &= (8+2+1) \oplus (16+4+2) \oplus (32+1) \\
&= (8+\cancel{2}+\cancel{1}) + (16+4+\cancel{2}) + (32+\cancel{1}) \\
&= 8+16+4+32 \\
&= 60.
\end{aligned}
$$

Proposition 1.3.5. *([1], p.112) The operation $\oplus$ is commutative and associative.*

Proof. This follows from the commutativity and associatively of $+$. ∎

Proposition 1.3.6. *For any two non-negative integers m and n, $m \oplus n \leq m + n$.*

Proof. This follows from the definition of Nim sum. ∎

Proposition 1.3.7. *([1],p.114) Given two impartial games G and H which correspond to Nim heaps of size g and h respectively, then their disjunctive sum $G + H$ corresponds to the Nim heap of size $g \oplus h$. That is, if $G = *_g$ and $H = *_h$, then $G + H = *_{g \oplus h}$.*

1.4 Normal versus Misère

Traditionally, combinatorial game theoreticians have concerned themselves with combinatorial games played under the normal play convention, although to a non-game theoretician, this choice may seem counterintuitive. Many of the first games to which we are introduced as children, although few are combinatorial games, are played under the misère game convention. Some examples are Crazy Eights, Snakes and Ladders, and Chinese Checkers, in which the goal is to either discard all one's cards or be first to the end of the board, leaving the winner with no moves left and the other players with moves still available. The reason for this choice is simply that misère games seem more difficult to analyse (as discussed in Section 1.1.7), and so the mathematical development of normal play games far surpasses that of misère games. Many simplistic ideas of how misère games should work in relation to normal games, such as if an impartial game is in $\mathcal{N}$ under normal play, then it must be in $\mathcal{P}$ under misère play (or vice versa), or a losing move under normal version of a game is a winning move under misère play, simply are not true. Example 1.1.4 showed that $\mathbb{2} + \mathbb{3} \in \mathcal{N}$ regardless of being played under the normal play or misère play convention. The following example shows that a bad move can be so under either normal or misère play.

Example 1.4.1. Consider the game $\mathbb{2} + \mathbb{2}$.

Suppose we are playing under the normal play convention, and Left's first move is to

$$\mathbb{2} + \mathbb{2} \xrightarrow{L} \mathbb{2}$$

i.e. taking one of the heaps of size two. Right responds in kind by taking the other heap of size two, leaving no tokens remaining. So Right wins under the normal play convention. However, Right also wins under the misère play convention if Left's first move is also to take one of the two heaps of size 2:

$$\mathbb{2} + \mathbb{2} \xrightarrow{L} \mathbb{2} \xrightarrow{R} \mathbb{1} \xrightarrow{L} \mathbb{0}$$

Therefore, a losing move under normal play does not necessarily translate into a winning move in misère play.

1.4.1 A Sprague-Grundy Theory for Impartial Misère Games?

Recall Theorem 1.3.2 - Every impartial game played under the normal play convention is equivalent to a Nim heap. Ideally, we would like to say the same for impartial games played under the misère play convention. However, as will be discussed in Section 2.2, this is not the case with impartial misère games. Much of the study of impartial misère games has been on determining which impartial misère games behave like Nim.

1.5 Thesis Overview

This rest of the thesis is divided as follows:

- Chapter 2 discusses the notion of **genus**, an important tool in impartial misère game theory. Knowledge of the genus of a position of a game not only tells us the outcome class for both the normal and misère play position, it also allows us to determine whether under the misère game convention, this game behaves like misère Nim.

- In Chapter 3, we use the genus to analyse subtraction games and octal games which do not permit splitting.
- In Chapter 4, we use the genus to analyse the game of Toppling Towers. This game has many similarities to octal games and allows both subtraction and splitting.
- In Chapter 5, we discuss the indistinguishablility quotient, a new method developed by Plambeck ([8]), which allows us to analyse a wide variety of "non-Nim" octal games.

Chapter 2

Genus

The traditional tool for impartial misère game analysis is the **genus symbol** or the **misère Grundy value**, developed by Conway ([4], Chapter 12 and [3], Chapter 13). The genus of an impartial misère game G allows one to determine whether this game is misère Nim in disguise and to calculate G's outcome class.

Although the statement of many of the results regarding genus have appeared in the literature, few of the results appear with proof. Statements which have appeared previously are referenced, although the proofs presented here are of the author's own construction.

Unless otherwise stated, all games in this chapter are impartial games.

2.1 Calculating the Genus

Definition. Fix a game G. We define

$$\mathcal{G}^+(G) = \begin{cases} 0 & \text{if } G \text{ has no options} \\ \operatorname{mex}\{\mathcal{G}^+(G') \mid G' \text{ is an option of } G\} & \text{else,} \end{cases}$$

and

$$\mathcal{G}^-(G) = \begin{cases} 1 & \text{if } G \text{ has no options} \\ \operatorname{mex}\{\mathcal{G}^-(G') \mid G' \text{ is an option of } G\} & \text{else.} \end{cases}$$

Note that $\mathcal{G}^+$ is the same as the value determined in Proposition 1.3.3. That is, for an impartial game G, $\mathcal{G}^+(G)$ corresponds to the Nim heap to which G is equivalent under the normal play convention.

Example 2.1.1. Consider $\mathbb{0}$. $\mathbb{0}$ has no options, so by definition:

$$\mathcal{G}^-(\mathbb{0}) \;=\; 1.$$

Consider $\mathbb{1}$. The moves available are:

$$\mathbb{1} \xrightarrow{-1} \mathbb{0}.$$

Therefore

$$\begin{aligned}\mathcal{G}^-(\mathbb{1}) &= \text{mex}\{\mathcal{G}^-(\mathbb{0})\}\\ &= \text{mex}\{1\}\\ &= 0\end{aligned}$$

Consider $\mathbb{2}$. The moves available are:

$$\begin{aligned}\mathbb{2} &\xrightarrow{-2} \mathbb{0}\\ &\xrightarrow{-1} \mathbb{1}\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{G}^-(\mathbb{2}) &= \text{mex}\{\mathcal{G}^-(\mathbb{0}), \mathcal{G}^-(\mathbb{1})\}\\ &= \text{mex}\{1, 0\}\\ &= 2\end{aligned}$$

Lemma 2.1.1. *For any game G, $\mathcal{G}^-(G + \mathbb{1}) = \mathcal{G}^-(G) \oplus 1$.*

Proof. We show this by induction on the options of G. Suppose G is a game with no options. Then $\mathcal{G}^-(G) = 1$.

Look now at $\mathcal{G}^-(G+\mathbb{1})$. Since we cannot move in G, we simply ignore its presence and consider the only available move in $\mathbb{1}$:

$$\begin{aligned}\mathcal{G}^-(G + \mathbb{1}) &= \text{mex}\{\mathcal{G}^-(\mathbb{0})\}\\ &= \text{mex}\{1\} \text{ since } \mathbb{0} \text{ has no options}\\ &= 0\\ &= 1 \oplus 1\\ &= \mathcal{G}^-(G) \oplus 1,\end{aligned}$$

which shows the base case.

Take a game G with options $G_0, G_1, \cdots, G_\mu$, and suppose the statement holds for all options of G. Then

$$\mathcal{G}^-(G+\mathbb{1}) = \text{mex}\{\mathcal{G}^-(G), \mathcal{G}^-(G_0+\mathbb{1}), \mathcal{G}^-(G_1+\mathbb{1}), \cdots, \mathcal{G}^-(G_\mu+\mathbb{1})\}.$$

By the induction hypothesis, we have that

$$\mathcal{G}^-(G+\mathbb{1}) = \text{mex}\{\mathcal{G}^-(G), \mathcal{G}^-(G_0) \oplus 1, \mathcal{G}^-(G_1) \oplus 1, \cdots, \mathcal{G}^-(G_\mu) \oplus 1\}.$$

Claim $\mathcal{G}^-(G+\mathbb{1}) = \mathcal{G}^-(G) \oplus 1$. We must show that no value in the mex set equals $\mathcal{G}^-(G) \oplus 1$ and there are values in the mex set equal to all non-negative integers less than $\mathcal{G}^-(G) \oplus 1$.

Suppose there exists an option of G, G_i, with $\mathcal{G}^-(G_i) \oplus 1 = \mathcal{G}^-(G) \oplus 1$. Then $\mathcal{G}^-(G_i) = \mathcal{G}^-(G)$, which is a contradiction.

Let $\mathcal{G}^-(G) = n$. Then, without loss of generality, we can assume that

$$\mathcal{G}^-(G_0) = 0, \mathcal{G}^-(G_1) = 1, \cdots, \mathcal{G}^-(G_{n-1}) = n-1,$$

and for $i \in \{n, n+1, \cdots, \mu\}$, $\mathcal{G}^-(G_i) \in \mathbb{N}$, $\mathcal{G}^-(G_i) \neq n$.

Then

$$\mathcal{G}^-(G) \oplus 1 = \begin{cases} n+1 & \text{if } n \equiv 0 \bmod(2) \\ n-1 & \text{if } n \equiv 1 \bmod(2). \end{cases}$$

Moreover

$$\begin{aligned}
\mathcal{G}^-(G_0) \oplus 1 &= 1, \\
\mathcal{G}^-(G_1) \oplus 1 &= 0, \\
\mathcal{G}^-(G_2) \oplus 1 &= 3, \\
\mathcal{G}^-(G_3) \oplus 1 &= 2, \\
&\cdots \\
\mathcal{G}^-(G_{n-4}) \oplus 1 &= \begin{cases} n-3 & \text{if } n \equiv 0 \bmod(2) \\ n-5 & \text{if } n \equiv 1 \bmod(2), \end{cases} \\
\mathcal{G}^-(G_{n-3}) \oplus 1 &= \begin{cases} n-4 & \text{if } n \equiv 0 \bmod(2) \\ n-2 & \text{if } n \equiv 1 \bmod(2), \end{cases} \\
\mathcal{G}^-(G_{n-2}) \oplus 1 &= \begin{cases} n-1 & \text{if } n \equiv 0 \bmod(2) \\ n-3 & \text{if } n \equiv 1 \bmod(2), \end{cases}
\end{aligned}$$

$$\mathcal{G}^-(G_{n-1}) \oplus 1 = \begin{cases} n-2 & \text{if } n \equiv 0 \bmod(2) \\ n & \text{if } n \equiv 1 \bmod(2). \end{cases}$$

Therefore for all $m < n \oplus 1$, there is a value in the mex set which equals m, and we have already shown that there does not exist $i \in \{1, 2, \cdots, \mu\}$ such that $\mathcal{G}^-(G_i) \oplus 1 = n$. Thus

$$\begin{aligned} \mathcal{G}^-(G + \mathbb{1}) &= \begin{cases} n+1 & \text{if } n \equiv 0 \bmod(2) \\ n-1 & \text{if } n \equiv 1 \bmod(2) \end{cases} \\ &= \mathcal{G}^-(G) \oplus 1, \end{aligned}$$

as required. ■

Definition. The **genus** of a misère game G, denoted by $\Gamma(G)$, is a list of the form $g^{g_0 g_1 g_2 g_3 \cdots}$ where

$$\begin{aligned} g &= \mathcal{G}^+(G), \\ g_0 &= \mathcal{G}^-(G), \\ g_1 &= \mathcal{G}^-(G + 2), \\ g_2 &= \mathcal{G}^-(G + 2 + 2), \\ &\cdots \\ g_n &= \mathcal{G}^-\left(G + \sum_{i=1}^{n} 2\right), \\ &\cdots \end{aligned}$$

Much like $\mathcal{G}^+$ and $\mathcal{G}^-$, the calculation $\Gamma(G)$ can be recursively determined by the genera of the options of G, as the following proposition shows.

Proposition 2.1.2. *([3], p.430) Suppose G is an impartial game with options G_a, G_b, G_c, $G_d, \cdots$ such that*

$$\begin{aligned} \Gamma(G_a) &= a^{a_0 a_1 a_2 a_3 \cdots}, \\ \Gamma(G_b) &= b^{b_0 b_1 b_2 b_3 \cdots}, \\ \Gamma(G_c) &= c^{c_0 c_1 c_2 c_3 \cdots}, \\ \Gamma(G_d) &= d^{d_0 d_1 d_2 d_3 \cdots}, \\ &\cdots \end{aligned}$$

Then $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$ *is calculated as follows:*

$$\begin{aligned}
g &= mex\{a, b, c, d \cdots\}, \\
g_0 &= mex\{a_0, b_0, c_0, d_0 \cdots\}, \\
g_1 &= mex\{g_0, g_0 \oplus 1, a_1, b_1, c_1, d_1 \cdots\}, \\
g_2 &= mex\{g_1, g_1 \oplus 1, a_2, b_2, c_2, d_2 \cdots\}, \\
&\cdots \\
g_n &= mex\{g_{n-1}, g_{n-1} \oplus 1, a_n, b_n, c_n, d_n \cdots\}.
\end{aligned}$$

Proof. We begin with g:

$$\begin{aligned}
g &= \operatorname{mex}\{\mathcal{G}^+(G_a), \mathcal{G}^+(G_b), \mathcal{G}^+(G_c), \mathcal{G}^+(G_d), \cdots\} \\
&= \operatorname{mex}\{a, b, c, d, \cdots\}.
\end{aligned}$$

Consider now g_0:

$$\begin{aligned}
g_0 &= \operatorname{mex}\{\mathcal{G}^-(G_a), \mathcal{G}^-(G_b), \mathcal{G}^-(G_c), \mathcal{G}^-(G_d), \cdots\} \\
&= \operatorname{mex}\{a_0, b_0, c_0, d_0, \cdots\}.
\end{aligned}$$

Fix $n \in \mathbb{N}$. We will show the result for g_n. The options of $G + \sum_{i=1}^{n} \mathit{2}$ are

$G + \sum_{i=1}^{n-1} \mathit{2}$,
$G + \sum_{i=1}^{n-1} \mathit{2} + \mathbb{1}$,
$G_j + \sum_{i=1}^{n} \mathit{2}$ for G_j an option of G.

Therefore

$$\begin{aligned}
g_n &= \operatorname{mex}\left\{\mathcal{G}^-\left(G + \sum_{i=1}^{n-1} \mathit{2}\right), \mathcal{G}^-\left(G + \sum_{i=1}^{n-1} \mathit{2} + \mathbb{1}\right),\right. \\
&\quad \left.\left\{\mathcal{G}^-\left(G_j + \sum_{i=1}^{n} \mathit{2}\right) \,\middle|\, G_j \text{ an option of } G\right\}\right\} \\
&= \operatorname{mex}\{g_{n-1}, g_{n-1} \oplus 1, \\
&\quad \left.\left\{\mathcal{G}^-\left(G_j + \sum_{i=1}^{n} \mathit{2}\right) \,\middle|\, G_j \text{ an option of } G\right\}\right\} \text{ by Lemma 2.1.1} \\
&= \operatorname{mex}\{g_{n-1}, g_{n-1} \oplus 1, a_n, b_n, c_n, d_n \cdots\},
\end{aligned}$$

which gives the result. ■

We are now able to calculate the genus of a game:

Proposition 2.1.3. *The genus of* $\mathbb{2}$ *is* $2^{2020202020202020\cdots}$. *That is, the superscript of the genus symbol alternates between 2 and 0.*

Proof. We begin with g:

$$\begin{aligned} g &= \mathcal{G}^{+}(\mathbb{2}) \\ &= \operatorname{mex}\{\mathcal{G}^{+}(\mathbb{0}), \mathcal{G}^{+}(\mathbb{1})\} \\ &= \operatorname{mex}\{0, \operatorname{mex}\{\mathcal{G}^{+}(\mathbb{0})\}\} \\ &= \operatorname{mex}\{0, \operatorname{mex}\{0\}\} \\ &= \operatorname{mex}\{0, 1\} \\ &= 2. \end{aligned}$$

Therefore $\Gamma(\mathbb{2}) = 2^{g_0 g_1 g_2 \cdots}$.

Claim that

$$g_n = \begin{cases} 2 & \text{if } n \equiv 0 \bmod(2) \\ 0 & \text{if } n \equiv 1 \bmod(2). \end{cases}$$

We will show this by induction on n.

$$\begin{aligned} g_0 &= \mathcal{G}^{-}(\mathbb{2}) \\ &= \operatorname{mex}\{\mathcal{G}^{-}(\mathbb{0}), \mathcal{G}^{-}(\mathbb{1})\} \\ &= \operatorname{mex}\{1, \operatorname{mex}\{\mathcal{G}^{-}(\mathbb{0})\}\} \\ &= \operatorname{mex}\{1, \operatorname{mex}\{1\}\} \\ &= \operatorname{mex}\{1, 0\} \\ &= 2 \end{aligned} \qquad \begin{aligned} g_1 &= \mathcal{G}^{-}(\mathbb{2} + \mathbb{2}) \\ &= \operatorname{mex}\{\mathcal{G}^{-}(\mathbb{2}), \mathcal{G}^{-}(\mathbb{2} + \mathbb{1})\} \\ &= \operatorname{mex}\{2, \mathcal{G}^{-}(\mathbb{2}) \oplus 1\} \text{ by Lemma 2.1.1} \\ &= \operatorname{mex}\{2, 2 \oplus 1\} \\ &= \operatorname{mex}\{2, 3\} \\ &= 0 \end{aligned}$$

This shows the base case.

Suppose true $\forall\, n < k$. Consider $n = k$.

$$\begin{aligned} g_k &= \mathcal{G}^{-}\left(\mathbb{2} + \sum_{i=1}^{k} \mathbb{2}\right) \\ &= \operatorname{mex}\left\{\mathcal{G}^{-}\left(\sum_{i=1}^{k} \mathbb{2}\right), \mathcal{G}^{-}\left(\sum_{i=1}^{k} \mathbb{2} + \mathbb{1}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{mex}\left\{\mathcal{G}^-\left(\sum_{i=1}^{k} 2\right), \mathcal{G}^-\left(\sum_{i=1}^{k} 2\right) \oplus 1\right\} \text{ by Lemma 2.1.1} \\
&= \operatorname{mex}\left\{\mathcal{G}^-\left(2 + \sum_{i=1}^{k-1} 2\right), \mathcal{G}^-\left(2 + \sum_{i=1}^{k-1} 2\right) \oplus 1\right\} \\
&= \operatorname{mex}\{g_{k-1}, g_{k-1} \oplus 1\}
\end{aligned}
$$

By the induction hypothesis:

$$
\begin{aligned}
g_{k-1} &= \begin{cases} 0 & \text{if } k \equiv 0 \bmod(2) \\ 2 & \text{if } k \equiv 1 \bmod(2), \end{cases} \\
g_{k-1} \oplus 1 &= \begin{cases} 1 & \text{if } k \equiv 0 \bmod(2) \\ 3 & \text{if } k \equiv 1 \bmod(2). \end{cases}
\end{aligned}
$$

Then

$$
\begin{aligned}
g_k &= \operatorname{mex}\{g_{k-1}, g_{k-1} \oplus 1\} \\
&= \begin{cases} \operatorname{mex}\{0,1\} & \text{if } k \equiv 0 \bmod(2) \\ \operatorname{mex}\{2,3\} & \text{if } k \equiv 1 \bmod(2) \end{cases} \\
&= \begin{cases} 2 & \text{if } k \equiv 0 \bmod(2) \\ 0 & \text{if } k \equiv 1 \bmod(2), \end{cases}
\end{aligned}
$$

which completes our proof. ■

This alternation leads to the following definition:

Definition. For a game G, we say that the genus of G, $g^{g_0 g_1 g_2 g_3 \cdots}$, **stabilises** if there exists an $N \in \mathbb{Z}^{\geq 0}$ such that $\forall\, n \geq N$,

$$g_{n+1} = g_n \oplus 2.$$

Lemma 2.1.4. *If $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$ has stabilised, then the digits in the superscript of $\Gamma(G)$ alternates between g_N and g_{N+1}.*

Proof. Suppose $\Gamma(G)$ stabilises at g_N for $N \in \mathbb{Z}^{\geq 0}$.

$$g_{N+1} = g_N \oplus 2.$$

Therefore

$$\begin{aligned} g_{N+2} &= g_{N+1} \oplus 2 \text{ by definition} \\ &= (g_N \oplus 2) \oplus 2 \text{ by definition} \\ &= g_N \oplus (2 \oplus 2) \\ &= g_N \oplus 0 \text{ by the rules of } \oplus \\ &= g_N. \end{aligned}$$

We also have

$$\begin{aligned} g_{N+3} &= g_{N+2} \oplus 2 \text{ by definition} \\ &= g_N \oplus 2 \text{ by previous calculations} \\ &= g_{N+1} \text{ by definition.} \end{aligned}$$

Continuing as such, we see that the digits in the superscript of $\Gamma(G)$ alternate between g_N and $g_N \oplus 2$. ■

Proposition 2.1.3 showed that *2* stabilises. In fact, this is true for every genus symbol, and is one of the key results of genus theory. We present a proof of the result, even though it is long and technical.

Theorem 2.1.5. *([3], p.422) Let G be an impartial game. Then the genus of G stabilises.*

Proof. We will show the following: for an impartial misère game G, there exists a non-negative integer N such that $\forall\ u \geq N$, $g_{u+1} = g_u \oplus 2$. Showing this then shows that for $u \geq N$, $g_u = g_{u+2}$ since if $g_{u+1} = g_u \oplus 2\ \forall\ u \geq N$:

$$\begin{aligned} g_{u+2} &= g_{u+1} \oplus 2 \\ &= (g_u \oplus 2) \oplus 2 \\ &= g_u \oplus 2 \oplus 2 \\ &= g_u. \end{aligned}$$

We proceed by induction on the options of the game.

Suppose G has no options. Then

$$\begin{aligned} g &= 0, \\ g_0 &= 1. \end{aligned}$$

Therefore $\Gamma(G) = 0^{1g_1g_2\cdots}$.

Look at g_n:

$$\begin{aligned} g_n &= \mathcal{G}^-\left(G + \sum_{i=1}^{n} 2\right) \\ &= \mathcal{G}^-\left(\sum_{i=1}^{n} 2\right) \\ &= \mathcal{G}^-\left(2 + \sum_{i=1}^{n-1} 2\right). \end{aligned}$$

Thus for $n \in \mathbb{N}$, g_n is the $n-1^{th}$ term in the superscript of $\Gamma(2)$. By Proposition 2.1.3,

$$g_n = \begin{cases} 2 & \text{if } n \equiv 1 \bmod(2) \\ 0 & \text{if } n \equiv 0 \bmod(2) \text{ and } n \geq 2. \end{cases}$$

and we see that for $n \geq 1$, $g_{n+1} = g_n \oplus 2$.

Suppose now we have a game G with options $G_0, G_1, \cdots, G_\mu$, where $\Gamma(G) = g^{g_0g_1g_2\cdots}$ and $\Gamma(G_i) = \gamma_i^{g_0^ig_1^ig_2^i\cdots}$ $\forall\ i \in \{0, 1, \cdots, \mu\}$ such that $G_0, G_1, \cdots, G_\mu$ satisfy the induction hypothesis. That is, for each G_i, there exists $n_i \in \mathbb{Z}^{\geq 0}$ such that $\forall\ u \geq n_i$,

$$g_{u+1}^i = g_u^i \oplus 2. \tag{2.1}$$

Let

$$n = \max\{n_i \mid i \in \{0, 1, \cdots, \mu\}\}.$$

Look at g_{n+1}. By Proposition 2.1.2,

$$g_{n+1} = \text{mex}\{g_n, g_n \oplus 1, g_{n+1}^0, g_{n+1}^1, \cdots, g_{n+1}^\mu\}.$$

Let $g_{n+1} = m$. Then there exists elements in the mex set of g_{n+1} equal to 0,1, $\cdots$, $m-1$. Basic calculations give that $\{g_n, g_n \oplus 1\} = \{t, t+1\}$ in some order for some $t \equiv 0 \bmod(2)$.

Look at g_{n+2}:

$$\begin{aligned} g_{n+2} &= \text{mex}\{g_{n+1}, g_{n+1} \oplus 1, g_{n+2}^0, g_{n+2}^1, \cdots, g_{n+2}^{\mu}\} \text{ by Proposition 2.1.2} \\ &= \text{mex}\{g_{n+1}, g_{n+1} \oplus 1, g_{n+1}^0 \oplus 2, g_{n+1}^1 \oplus 2, \cdots, g_{n+1}^{\mu} \oplus 2\} \text{ by Equation (2.1).} \end{aligned}$$

We now break the proof into cases. Although the method of proof is similar for all cases, we include the proofs for completeness.

We first consider two cases: $t > m$ and $t < m$. We do not consider $t = m$ since $t = g_n$ is in the mex set of $g_{n+1} = m$.

1) Suppose $t > m$.

Then, we can assume without loss of generality, that

$$\begin{aligned} g_{n+1}^0 &= 0, \\ g_{n+1}^1 &= 1, \\ g_{n+1}^2 &= 2, \\ &\cdots \\ g_{n+1}^{m-2} &= m-2, \\ g_{n+1}^{m-1} &= m-1, \\ g_{n+1}^i &\neq m \text{ for } i \in \{m, m+1, \cdots, \mu\}. \end{aligned}$$

Claim $g_{n+2} = m \oplus 2$.

Suppose $g_{n+1}^j \oplus 2 = m \oplus 2$ for some $j \in \{0, 1, \cdots, \mu\}$. Then $g_{n+1}^j = m$, which gives a contradiction since $g_{n+1} = m$.

Calculating and reordering the values in the mex, we have

$$g_{n+2} = \begin{cases} \text{mex}\{0, 1, 2, \cdots, m-2, m-1, m, m+1, g_{n+1}^m \oplus 2, \cdots, g_{n+1}^{\mu} \oplus 2\} \\ \hfill \text{if } m \equiv 0, 1 \bmod(4) \\ \text{mex}\{0, 1, 2, \cdots, m-4, m-3, m-1, m, m+1, \\ \quad g_{n+1}^m \oplus 2, \cdots, g_{n+1}^{\mu} \oplus 2\} \\ \hfill \text{if } m \equiv 2 \bmod(4) \\ \text{mex}\{0, 1, 2, \cdots, m-4, m-3, m-1, m, g_{n+1}^m \oplus 2, \cdots, g_{n+1}^{\mu} \oplus 2\} \\ \hfill \text{if } m \equiv 3 \bmod(4) \end{cases}$$

$$
\begin{aligned}
&= \begin{cases} m+2 & \text{if } m \equiv 0,1 \bmod(4) \\ m-2 & \text{if } m \equiv 2,3 \bmod(4) \end{cases} \\
&= m \oplus 2 \\
&= g_{n+1} \oplus 2.
\end{aligned}
$$

2) Suppose that $t < m$.

Then, we can assume without loss of generality, that

$$
\begin{aligned}
g_{n+1}^{0} &= 0, \\
g_{n+1}^{1} &= 1, \\
g_{n+1}^{2} &= 2, \\
&\cdots \\
g_{n+1}^{t-1} &= t-1. \\
-- \quad &-- \quad -- \\
g_{n+1}^{t+2} &= t+2, \\
&\cdots \\
g_{n+1}^{m-2} &= m-2, \\
g_{n+1}^{m-1} &= m-1, \\
g_{n+1}^{i} &\neq m \text{ for } i \in \{t, t+1, m, m+1, \cdots, \mu\}.
\end{aligned}
$$

We must now further decompose the proof into ten disjoint cases: (a) through (j). Again, although the method of proof is similar in each case, we include all but the proof of (j) for completeness.

a) Suppose $g_{n+1}^{m}, \cdots, g_{n+1}^{\mu}, g_{n+1}^{t}, g_{n+1}^{t+1} \in \mathbb{Z}^{\geq 0} \setminus \{t, t+1, m\}$ and $t+2 \neq m, m \oplus 1$.

Claim $g_{n+2} = t \oplus 2$.

Suppose $g_{n+1}^{j} \oplus 2 = t \oplus 2$ for some $j \in \{0, 1, \cdots, \mu\}$. Then $g_{n+1}^{j} = t$, but $g_{n+1}^{j} \in \mathbb{Z}^{\geq 0} \setminus \{t, t+1, m\}$, which yields a contradiction.

Calculating and reordering the values in the mex, we have

$$g_{n+2} = \begin{cases} \operatorname{mex}\{0,1,2,\cdots,t-2,t-1,t,t+1,t+4,\cdots(m-1)\oplus 2, \\ \quad g^t_{n+1}\oplus 2, g^{t+1}_{n+1}\oplus 2, g^m_{n+1}\oplus 2,\cdots,g^\mu_{n+1}\oplus 2, m, m\oplus 1\} \\ \quad \text{if } t\equiv 0 \bmod(4) \\ \operatorname{mex}\{0,1,2,\cdots,t-4,t-3,t,t+1,t+2,\cdots(m-1)\oplus 2, \\ \quad g^t_{n+1}\oplus 2, g^{t+1}_{n+1}\oplus 2, g^m_{n+1}\oplus 2,\cdots,g^\mu_{n+1}\oplus 2, m, m\oplus 1\} \\ \quad \text{if } t\equiv 2 \bmod(4) \end{cases}$$
$$= \begin{cases} t+2 & \text{if } t\equiv 0 \bmod(4) \\ t-2 & \text{if } t\equiv 2 \bmod(4) \end{cases}$$
$$= t\oplus 2.$$

Examine g_{n+3}:

$$g_{n+3} = \begin{cases} \operatorname{mex}\{t+2,t+3,0,1,\cdots,t-1,t+2,\cdots(m-1), \\ \quad g^t_{n+1}, g^{t+1}_{n+1}, g^m_{n+1},\cdots,g^\mu_{n+1}\} \text{ if } t\equiv 0 \bmod(4) \\ \operatorname{mex}\{t-2,t-1,0,1,\cdots,t-1,t+2,\cdots(m-1), \\ \quad g^t_{n+1}, g^{t+1}_{n+1}, g^m_{n+1},\cdots,g^\mu_{n+1}\} \text{ if } t\equiv 2 \bmod(4) \end{cases}$$
$$= t$$
$$= (t\oplus 2)\oplus 2$$
$$= g_{n+2}\oplus 2.$$

b) Suppose $g^m_{n+1},\cdots,g^\mu_{n+1},g^t_{n+1},g^{t+1}_{n+1}\in\mathbb{Z}^{\geq 0}\setminus\{t,t+1,m\}$ and $t+2=m$.

By Case (2a), this will only affect the calculations when $t\equiv 0 \bmod(4)$. Take $t\equiv 0 \bmod(4)$.

Since $m=t+2$,

$$\begin{aligned} g^0_{n+1} &= 0, \\ g^1_{n+1} &= 1, \\ g^2_{n+1} &= 2, \\ &\cdots \\ g^{t-2}_{n+1} &= t-2, \\ g^{t-1}_{n+1} &= t-1, \\ g^i_{n+1} &\neq m,t,t+1 \text{ for } i\in\{t,t+1,\cdots,\mu\}. \end{aligned}$$

Claim $g_{n+2} = t$.

Suppose $g^j_{n+1} \oplus 2 = t$. Since $t \equiv 0 \bmod (4)$, this implies that $g^j_{n+1} = t+2$, but $m = t+2$, which yields a contradiction.

Calculating and reordering the values in the mex, we have

$$\begin{aligned} g_{n+2} &= \text{mex}\{0, 1, 2, \cdots, t-2, t-1, t+2, t+3, g^t_{n+1} \oplus 2, g^{t+1}_{n+1} \oplus 2, \\ &\quad g^m_{n+1} \oplus 2, \cdots, g^\mu_{n+1} \oplus 2\} \\ &= t. \end{aligned}$$

Examine g_{n+3}:

$$\begin{aligned} g_{n+3} &= \text{mex}\{t, t+1, 0, 1, \cdots, t-1, g^t_{n+1}, g^{t+1}_{n+1}, g^m_{n+1}, \cdots, g^\mu_{n+1}\} \\ &= t+2 \\ &= t \oplus 2 \text{ since } t \equiv 0 \bmod(4) \\ &= g_{n+2} \oplus 2. \end{aligned}$$

c) Suppose $g^m_{n+1}, \cdots, g^\mu_{n+1}, g^t_{n+1}, g^{t+1}_{n+1} \in \mathbb{Z}^{\geq 0} \setminus \{t, t+1, m\}$ and $t+2 = m \oplus 1$.

By Case (2a), this will only affect the calculations when $t \equiv 0 \bmod(4)$. Take $t \equiv 0 \bmod(4)$.

Since $m \oplus 1 = t+2$ and $t \equiv 0 \bmod(4)$, this implies that $m = t+3$. Then,

$$\begin{aligned} g^0_{n+1} &= 0, \\ g^1_{n+1} &= 1, \\ g^2_{n+1} &= 2, \\ &\cdots \\ g^{t-2}_{n+1} &= t-2, \\ g^{t-1}_{n+1} &= t-1. \\ -- \quad &-- \quad -- \\ g^{t+2}_{n+1} &= t+2, \\ g^i_{n+1} &\neq m, t, t+1 \text{ for } i \in \{t, t+1, t+3, \cdots, \mu\}\ . \end{aligned}$$

Claim $g_{n+2} = t+1$.

Suppose $g^j_{n+1} \oplus 2 = t+1$. Since $t \equiv 0 \bmod(4)$, this implies that $g^j_{n+1} = t+3$. But $m = t+3$, which gives a contradiction.

Calculating and reordering the values in the mex, we have

$$\begin{aligned} g_{n+2} &= \operatorname{mex}\{0,1,2,\cdots,t-2,t-1,t,t+2,t+3,g^t_{n+1}\oplus 2, g^{t+1}_{n+1}\oplus 2, \\ &\quad g^m_{n+1}\oplus 2,\cdots,g^\mu_{n+1}\oplus 2\} \\ &= t+1. \end{aligned}$$

Examine g_{n+3}:

$$\begin{aligned} g_{n+3} &= \operatorname{mex}\{t+1,t,0,1,2,\cdots,t-1,t+2,g^t_{n+1},g^{t+1}_{n+1},g^m_{n+1},\cdots,g^\mu_{n+1}\} \\ &= t+3 \\ &= (t+1)\oplus 2 \text{ since } t \equiv 0 \bmod (4) \\ &= g_{n+2}\oplus 2. \end{aligned}$$

d) Suppose $g^t_{n+1} = t$, $g^m_{n+1},\cdots,g^\mu_{n+1},g^{t+1}_{n+1} \in \mathbb{Z}^{\geq 0}\setminus\{t+1,m\}$, and $t+3 \neq m, m\oplus 1$.

Claim $g_{n+2} = (t\oplus 2)+1$.

Suppose $g^j_{n+1}\oplus 2 = (t\oplus 2)+1$ for some $j \in \{0,1,\cdots,\mu\}$. Then $g^j_{n+1} = t+1$, which yields a contradiction.

This case is similar to Case (2a), except that $t\oplus 2$ is added to the mex set of g_{n+2}:

$$t\oplus 2 = \begin{cases} t+2 & \text{if } t\equiv 0 \bmod(4) \\ t-2 & \text{if } t \equiv 2 \bmod(4). \end{cases}$$

Calculating and reordering the values in the mex,

$$\begin{aligned} g_{n+2} &= \begin{cases} \operatorname{mex}\{0,1,2,\cdots,t-2,t-1,t,t+1,t+2,t+4,\cdots(m-1)\oplus 2, \\ \quad g^t_{n+1}\oplus 2, g^{t+1}_{n+1}\oplus 2, g^m_{n+1}\oplus 2,\cdots,g^\mu_{n+1}\oplus 2,m,m\oplus 1\} \\ \quad \text{if } t\equiv 0 \bmod(4) \\ \operatorname{mex}\{0,1,2,\cdots,t-4,t-3,t-2,t,t+1,t+2,\cdots(m-1)\oplus 2, \\ \quad g^t_{n+1}\oplus 2, g^{t+1}_{n+1}\oplus 2, g^m_{n+1}\oplus 2,\cdots,g^\mu_{n+1}\oplus 2,m,m\oplus 1\} \\ \quad \text{if } t\equiv 2 \bmod(4) \end{cases} \\ &= \begin{cases} t+3 & \text{if } t\equiv 0 \bmod(4) \\ t-1 & \text{if } t\equiv 2 \bmod(4) \end{cases} \\ &= (t\oplus 2)+1. \end{aligned}$$

Then

$$g_{n+3} = \begin{cases} \text{mex}\{t+3, t+2, 0, 1, \cdots, t-1, t, t+2, \cdots (m-1), \\ \quad g^t_{n+1}, g^{t+1}_{n+1}, g^m_{n+1}, \cdots, g^\mu_{n+1}\} \text{ if } t \equiv 0 \text{ mod}(4) \\ \text{mex}\{t-1, t-2, 0, 1, \cdots, t-1, t, t+2, \cdots (m-1), \\ \quad g^t_{n+1}, g^{t+1}_{n+1}, g^m_{n+1}, \cdots, g^\mu_{n+1}\} \text{ if } t \equiv 0 \text{ mod}(4) \end{cases}$$

$$= t+1$$

$$= ((t \oplus 2) + 1) \oplus 2$$

$$= g_{n+2} \oplus 2.$$

e) Suppose $g^t_{n+1} = t$, $g^m_{n+1}, \cdots, g^\mu_{n+1}, g^{t+1}_{n+1} \in \mathbb{Z}^{\geq 0} \setminus \{t+1, m\}$, and $t+3 = m$.

By Case (2d), this will only affect the calculations when $t \equiv 0$ mod(4). Take $t \equiv 0$ mod(4).

Since $m = t+3$, $m \oplus 1 = t+2$. Then

$$\begin{aligned} g^0_{n+1} &= 0, \\ g^1_{n+1} &= 1, \\ g^2_{n+1} &= 2, \\ &\cdots \\ g^{t-1}_{n+1} &= t-1, \\ g^t_{n+1} &= t. \\ -- \quad &-- \quad -- \\ g^{t+2}_{n+1} &= t+2, \\ g^i_{n+1} &\neq m, t+1 \text{ for } i \in \{t+1, t+3, t+4, \cdots \mu\}. \end{aligned}$$

Claim $g_{n+2} = t+1$.

Suppose $g^j_{n+1} \oplus 2 = t+1$. Since $t \equiv 0$ mod (4), this implies that $g^j_{n+1} = t+3$, but $m = t+3$, which yields a contradiction.

Calculating and reordering the values in the mex, we have

$$g_{n+2} = \text{mex}\{0, 1, 2, \cdots, t-1, t, t+2, t+3, g^{t+1}_{n+1} \oplus 2, g^m_{n+1} \oplus 2, \cdots, g^\mu_{n+1} \oplus 2\}$$

$$= t+1.$$

Examine g_{n+3}:

$$\begin{aligned} g_{n+3} &= \operatorname{mex}\{t+1, t, 0, 1, \cdots, t-1, t, t+2, g_{n+1}^{t+1}, g_{n+1}^{m}, \cdots, g_{n+1}^{\mu}\} \\ &= t+3 \\ &= (t+1) \oplus 2 \text{ since } t \equiv 0 \bmod(4) \\ &= g_{n+2} \oplus 2. \end{aligned}$$

f) Suppose $g_{n+1}^t = t$, $g_{n+1}^m, \cdots, g_{n+1}^{\mu}, g_{n+1}^{t+1} \in \mathbb{Z}^{\geq 0} \setminus \{t+1, m\}$, and $t+3 = m \oplus 1$.

By Case (2d), this will only affect the calculations when $t \equiv 0 \bmod(4)$. Take $t \equiv 0 \bmod(4)$.

Since $m \oplus 1 = t+3$, $m = t+2$. Then

$$\begin{aligned} g_{n+1}^0 &= 0, \\ g_{n+1}^1 &= 1, \\ g_{n+1}^2 &= 2, \\ &\cdots \\ g_{n+1}^{t-1} &= t-1, \\ g_{n+1}^t &= t, \\ g_{n+1}^i &\neq m, t+1 \text{ for } i \in \{t+1, t+2, \cdots, \mu\}. \end{aligned}$$

Claim $g_{n+2} = t$.

Suppose $g_{n+1}^j \oplus 2 = t$. Since $t \equiv 0 \bmod (4)$, this implies that $g_{n+1}^j = t+2$, but $m = t+2$, which yields a contradiction.

Calculating and reordering the values in the mex, we have

$$\begin{aligned} g_{n+2} &= \operatorname{mex}\{0, 1, 2, \cdots, t-2, t-1, t+2, t+3, g_{n+1}^{t+1} \oplus 2, g_{n+1}^{m} \oplus 2, \\ &\quad \cdots, g_{n+1}^{\mu} \oplus 2\} \\ &= t. \end{aligned}$$

Examine g_{n+3}:

$$g_{n+3} = \operatorname{mex}\{t, t+1, 0, 1, \cdots, t-1, t, g_{n+1}^{t+1}, g_{n+1}^{m}, \cdots, g_{n+1}^{\mu}\}$$

$$
\begin{aligned}
&= t+2 \\
&= t \oplus 2 \text{ since } t \equiv 0 \bmod(4) \\
&= g_{n+2} \oplus 2.
\end{aligned}
$$

g) Suppose $g_{n+1}^{t+1} = t+1$, $g_{n+1}^{t}, g_{n+1}^{m}, \cdots, g_{n+1}^{\mu} \in \mathbb{Z}^{\geq 0} \setminus \{t, m\}$, and $m, m \oplus 1 \neq t+2$.

Claim $g_{n+2} = t \oplus 2$.

Suppose $g_{n+1}^{j} \oplus 2 = t \oplus 2$ for some $j \in \{0, 1, \cdots, \mu\}$. Then $g_{n+1}^{j} = t$, but $g_{n+1}^{j} \in \mathbb{Z}^{\geq 0} \setminus \{t, m\}$, which yields a contradiction.

This case is similar to Case (2a), except that $(t+1) \oplus 2$ is added to the mex set of g_{n+2}:

$$
(t+1) \oplus 2 = \begin{cases} t+3 & \text{if } t \equiv 0 \bmod(4) \\ t-1 & \text{if } t \equiv 2 \bmod(4). \end{cases}
$$

Calculating and reordering the values in the mex, we have

$$
\begin{aligned}
g_{n+2} &= \begin{cases} \operatorname{mex}\{0, 1, 2, \cdots, , t-1, t, t+1, t+3, \cdots (m-1) \oplus 2, \\ \quad g_{n+1}^{t} \oplus 2, g_{n+1}^{m} \oplus 2, \cdots, g_{n+1}^{\mu} \oplus 2, m, m \oplus 1\} \text{ if } t \equiv 0 \bmod(4) \\ \operatorname{mex}\{0, 1, 2, \cdots, t-3, t-1, t, t+1, t+2, \cdots (m-1) \oplus 2, \\ \quad g_{n+1}^{t} \oplus 2, g_{n+1}^{m} \oplus 2, \cdots, g_{n+1}^{\mu} \oplus 2, m, m \oplus 1\} \text{ if } t \equiv 2 \bmod(4) \end{cases} \\
&= \begin{cases} t+2 & \text{if } t \equiv 0 \bmod(4) \\ t-2 & \text{if } t \equiv 2 \bmod(4) \end{cases} \\
&= t \oplus 2.
\end{aligned}
$$

Examine g_{n+3}:

$$
\begin{aligned}
g_{n+3} &= \begin{cases} \operatorname{mex}\{t+2, t+3, 0, 1, \cdots, t-1, t+1, \cdots (m-1), \\ \quad g_{n+1}^{t}, g_{n+1}^{t+1}, g_{n+1}^{m}, \cdots, g_{n+1}^{\mu}\} \text{ if } t \equiv 0 \bmod(4) \\ \operatorname{mex}\{t-2, t-1, 0, 1, \cdots, t-1, t+1, \cdots (m-1), \\ \quad g_{n+1}^{t}, g_{n+1}^{t+1}, g_{n+1}^{m}, \cdots, g_{n+1}^{\mu}\} \text{ if } t \equiv 0 \bmod(4) \end{cases} \\
&= t \\
&= (t \oplus 2) \oplus 2 \\
&= g_{n+2} \oplus 2.
\end{aligned}
$$

h) Suppose $g_{n+1}^{t+1} = t+1$, $g_{n+1}^{t}, g_{n+1}^{m}, \cdots, g_{n+1}^{\mu} \in \mathbb{Z}^{\geq 0} \setminus \{t, m\}$, and $m = t+2$.

By Case (2g), this will only affect the calculations when $t \equiv 0$ mod(4). Take $t \equiv 0$ mod(4).

Since $m = t + 2$,

$$\begin{aligned}
g_{n+1}^{0} &= 0, \\
g_{n+1}^{1} &= 1, \\
g_{n+1}^{2} &= 2, \\
&\cdots \\
g_{n+1}^{t-1} &= t-1. \\
-- \quad -- & \quad -- \\
g_{n+1}^{t+1} &= t+1, \\
g_{n+1}^{i} &\neq m, t \text{ for } i \in \{t, t+2, t+3, \cdots, \mu\}.
\end{aligned}$$

Claim $g_{n+2} = t$.

Suppose $g_{n+1}^{j} \oplus 2 = t$. Since $t \equiv 0$ mod (4), this implies that $g_{n+1}^{j} = t + 2$, but $m = t + 2$, which yields a contradiction.

Calculating and reordering the values in the mex, we have

$$\begin{aligned}
g_{n+2} &= \operatorname{mex}\{0, 1, 2, \cdots, t-2, t-1, t+2, t+3, g_{n+1}^{t} \oplus 2, g_{n+1}^{m} \oplus 2, \\
&\quad \cdots, g_{n+1}^{\mu} \oplus 2\} \\
&= t.
\end{aligned}$$

Examine g_{n+3}:

$$\begin{aligned}
g_{n+3} &= \operatorname{mex}\{t, t+1, 0, 1, \cdots, t-1, t+1, g_{n+1}^{t}, g_{n+1}^{m}, \cdots, g_{n+1}^{\mu}\} \\
&= t+2 \\
&= t \oplus 2 \text{ since } t \equiv 0 \text{ mod}(4) \\
&= g_{n+2} \oplus 2.
\end{aligned}$$

i) Suppose $g_{n+1}^{t+1} = t + 1$, $g_{n+1}^{t}, g_{n+1}^{m}, \cdots, g_{n+1}^{\mu} \in \mathbb{Z}^{\geq 0} \setminus \{t, m\}$, and $m \oplus 1 = t + 2$.

By Case (2g), this will only affect the calculations when $t \equiv 0$ mod(4). Take $t \equiv 0$ mod(4).

Since $m \oplus 1 = t+2$, $m = t+3$. Then

$$\begin{array}{rcl}
g_{n+1}^{0} & = & 0, \\
g_{n+1}^{1} & = & 1, \\
g_{n+1}^{2} & = & 2, \\
 & \cdots & \\
g_{n+1}^{t-1} & = & t-1. \\
-- & -- & -- \\
g_{n+1}^{t+1} & = & t+1, \\
g_{n+1}^{t+2} & = & t+2, \\
g_{n+1}^{i} & \neq & m, t \text{ for } i \in \{t, t+3, t+4, \cdots, \mu\}.
\end{array}$$

Claim $g_{n+2} = t+1$.

Suppose $g_{n+1}^{j} \oplus 2 = t+1$. Since $t \equiv 0 \bmod (4)$, this implies that $g_{n+1}^{j} = t+3$, but $m = t+3$, which yields a contradiction.

Calculating and reordering the values in the mex, we have

$$\begin{aligned}
g_{n+2} &= \text{mex}\{0, 1, 2, \cdots, t-1, t, t+2, t+3, g_{n+1}^{t} \oplus 2, g_{n+1}^{m} \oplus 2, \cdots, g_{n+1}^{\mu} \oplus 2\} \\
&= t+1.
\end{aligned}$$

Examine g_{n+3}:

$$\begin{aligned}
g_{n+3} &= \text{mex}\{t+1, t, 0, 1, \cdots, t-1, t+1, t+2, g_{n+1}^{t}, g_{n+1}^{m}, \cdots, g_{n+1}^{\mu}\} \\
&= t+3 \\
&= (t+1) \oplus 2 \text{ since } t \equiv 0 \bmod(4) \\
&= g_{n+2} \oplus 2.
\end{aligned}$$

j) Suppose $g_{n+1}^{t} = t$, $g_{n+1}^{t+1} = t+1$, $g_{n+1}^{m}, \cdots, g_{n+1}^{\mu} \in \mathbb{Z}^{\geq 0} \setminus \{m\}$.

The proof of this is the same as the proof of Case (1).

Thus, for all permutations of g_n, $g_n \oplus 1$, and g_{n+1}^{i}, we have found an index N such that $g_{N+1} = g_N \oplus 2$.

Claim: $\forall\ u \geq N,\ u \in \mathbb{N},\ g_{u+1} = g_u \oplus 2$. We will show this by induction on u. $u = N$ is precisely what was just shown. $u = N+1$ is shown similarly.

Suppose that $\forall\ u \in \mathbb{N}$ such that $N \leq u < k$,

$$g_{u+1} = g_u \oplus 2. \tag{2.2}$$

Suppose $u = k$. Then

$$\begin{aligned}
g_{k+1} &= \operatorname{mex}\{g_k, g_k \oplus 1, g^0_{k+1}, g^1_{k+1}, \cdots, g^\mu_{k+1}\} \\
&= \operatorname{mex}\{g_{k-1} \oplus 2, (g_{k-1} \oplus 2) \oplus 1, g^0_{k+1}, g^1_{k+1}, \cdots, g^\mu_{k+1}\} \text{ by Equation (2.2)} \\
&= \operatorname{mex}\{g_{k-2}, g_{k-2} \oplus 1, g^0_{k+1}, g^1_{k+1}, \cdots, g^\mu_{k+1}\} \text{ by Equation (2.2)} \\
&= \operatorname{mex}\{g_{k-2}, g_{k-2} \oplus 1, g^0_k \oplus 2, g^1_k \oplus 2, \cdots, g^\mu_k \oplus 2\} \text{ by Equation (2.1)} \\
&= \operatorname{mex}\{g_{k-2}, g_{k-2} \oplus 1, g^0_{k-1}, g^1_{k-1}, \cdots, g^\mu_{k-1}\} \text{ by Equation (2.1)} \\
&= g_{k-1} \text{ by Proposition 2.1.2} \\
&= g_k \oplus 2 \text{ by Equation (2.2).}
\end{aligned}$$

This completes both inductions. Therefore the genus of any game eventually stabilises. ■

Notation. Consider an impartial game G with $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$. By Theorem 2.1.5, G stabilises. Therefore there exists a smallest non-negative integer N such that $\forall\ u \geq N,\ g_u = g_{u+2}$ with $g_{u+1} = g_u \oplus 2$. We abbreviate the genus symbol to $g^{g_0 g_1 \cdots g_N (g_N \oplus 2)}$.

Corollary 2.1.6. *Given a game G with $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$, let N be the first index at which $g_{N+1} = g_N \oplus 2$. Then $\forall\ u \geq N,\ g_u = g_{u+2}$ provided the genus of all the options of G has stabilised by N.*

The corollary, which follows from the inductions in the previous proof, gives us the following calculational shortcut: once the genera of all the options of G have stabilised and G is exhibiting stabilizing behaviour, we can stop calculating the genus of G. It is not enough to stop calculations simply when the genus exhibits stabilizing behaviour, as the following example shows:

Example 2.1.2. Consider the octal game 0.3122. That is:

1) We can remove one token no matter the size of the heap.

2) We can remove two tokens if the heap has size two.

3) We can remove three tokens if the heap has strictly more than three tokens.

4) We can remove four tokens if the heap has strictly more than four tokens.

5) We cannot remove n tokens for $n \geq 5$.

Calculations give the genera of the first six heaps:

heap	genus
h_0	0^{120}
h_1	1^{031}
h_2	2^{20}
h_3	0^{02}
h_4	2^{1420}
h_5	0^{31}
h_6	1^{13}

We now calculate the genus of h_7. Initial calculations show us that $\Gamma(h_7) = 3^{20g_2g_3\cdots}$, and we see that $\mathcal{G}^-(h_7 + 2) = \mathcal{G}^-(h_7) \oplus 2$. However, the genus of an option of h_7, namely, $\Gamma(h_4)$, has not yet stabilised, and further calculations show that $\Gamma(h_7) = 3^{2031}$.

2.2 Using the Genus to Classify Impartial Misère Games

We begin by examining misère Nim:

Proposition 2.2.1. *Given a Nim heap* $\mathbb{m}$,

$$\Gamma(\mathbb{m}) = \begin{cases} 0^{120} & \text{if } m = 0 \\ 1^{031} & \text{if } m = 1 \\ m^{m(m \oplus 2)} & \text{else.} \end{cases}$$

Proof. When $m = 0$, $\Gamma(\mathbb{m}) = 0^{120}$ by the base case of Theorem 2.1.5.

Examine $m = 1$. There is only one move:

$$\mathbb{1} \longrightarrow \mathbb{0}.$$

Then, by Proposition 2.1.2,

$$\begin{aligned}
\mathcal{G}^{+}(\mathbb{1}) &= \operatorname{mex}\{\mathcal{G}^{+}(\mathbb{0})\} \\
&= \operatorname{mex}\{0\} \\
&= 1, \\
\mathcal{G}^{-}(\mathbb{1}) &= \operatorname{mex}\{\mathcal{G}^{-}(\mathbb{0}\} \\
&= \operatorname{mex}\{1\} \\
&= 0, \\
\mathcal{G}^{-}(\mathbb{1}+\mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^{-}(\mathbb{2}), \mathcal{G}^{-}(\mathbb{1}), \mathcal{G}^{-}(\mathbb{1}) \oplus 1\} \\
&= \operatorname{mex}\{2, 1, 0\} \\
&= 3, \\
\mathcal{G}^{-}(\mathbb{1}+\mathbb{2}+\mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^{-}(\mathbb{2}+\mathbb{2}), \mathcal{G}^{-}(\mathbb{1}+\mathbb{2}), \mathcal{G}^{-}(\mathbb{1}+\mathbb{2}) \oplus 1\} \\
&= \operatorname{mex}\{0, 3, 2\} \\
&= 1.
\end{aligned}$$

Therefore, $\Gamma(\mathbb{1}) = 1^{031}$.

For $m \geq 2$, we proceed by induction on m.

When $m = 2$, $\Gamma(\mathbb{m}) = 2^{20}$ by Proposition 2.1.3.

Suppose true for $m < k$. Consider $\mathbb{k}$. The moves from $\mathbb{k}$ are

$$\begin{array}{lcl}
\mathbb{k} & \xrightarrow{-k} & \mathbb{0} \\
\mathbb{k} & \xrightarrow{-k+1} & \mathbb{1} \\
\mathbb{k} & \xrightarrow{-k+2} & \mathbb{2} \\
\cdots & & \\
\mathbb{k} & \xrightarrow{-2} & \mathbb{k}-\mathbb{2} \\
\mathbb{k} & \xrightarrow{-1} & \mathbb{k}-\mathbb{1}.
\end{array}$$

By induction, the genera of the options are 0^{120}, 1^{031}, and $i^{i(i\oplus 2)}$ for $i \in \{2, 3, \cdots, k-1\}$ respectively. By Proposition 2.1.2, $\Gamma(\Bbbk) = k^{k(k\oplus 2)}$, as required. ■

Impartial misère games are classified into two types: tame and wild. The genus is the tool necessary to make this classification. As under the normal play convention, Nim plays a vital role:

Definition. Given an impartial misère game G, G is **tame** if $\Gamma(G)$ is one of the misère Nim genus symbols given in Proposition 2.2.1, 0^{02}, or 1^{13}, and, for every option G' of G, G' is tame. An impartial misère game is **wild** if it is not tame.

Definition. If a game $\Gamma(G)$ equals one of the misère Nim genus symbols given in Proposition 2.2.1, 0^{02}, or 1^{13}, but we do not know anything about its options, we say that $\Gamma(G)$ has a **tame value**.

There is a difference between being tame and have a tame value. We can only determine whether a game is tame if we know about its options. A tame game has a tame value, but the reverse is not necessarily true. A game can have a tame value but have a wild game as one of its followers.

Note: Some sources (notably [3], p.425) allow a game to be called tame even if it has wild positions in its options, provided that these wild positions satisfy certain conditions. However, we are taking the convention of [4] and taking that a tame game "behaves", for all positions, exactly like Nim. Thus, if in the followers of a position G, there is a wild position, then we consider G to be wild.

If a game is tame, we say that under the misère play convention, this game behaves like misère Nim. Given a game G which is the finite disjunctive sum of Nim heaps, if we play in this game, we play to a disjunctive sum of Nim heaps. Playing under the misère play convention, $\Gamma(G)$ has a tame value, as do all of its followers, i.e. it is tame. If another impartial game which has this property, then it is behaving like the way a disjunctive sum of Nim heaps behaves and so it is behaving the way misère Nim does. However, if an impartial game does not have its genus equal to a tame value, or has wild games as its options, which never occurs with a disjunctive sum of Nim heaps, then it is not behaving the way misère Nim behaves.

Under the normal play convention, all impartial games behave like Nim. That is, under normal play, every game is "normally tame" and there are no "normally wild" games. The initial hope that we can easily translate the Sprague-Grundy Theorem for normal play impartial games to misère impartial games would require that all impartial misère game be tame. Sadly, this is not the case as there is a wealth of examples of impartial misère games which are not tame. One such game is discussed in the following example:

Example 2.2.1. Consider a heap of size 8 in the octal game 0.123. The genus symbol of this game is 2^{1420} and so a heap of size 8 in 0.123 is wild, and so, the game 0.123 is wild.

2.2.1 Getting Wild Games from Tame Games

A standard trick in game theory is the following - given a property P, and a game G, if all options of G have property P, then show that G also has property P. This is exactly how we proved Theorem 2.1.5.

However, we cannot use this method in determining whether a position is tame. That is, all options of a position being tame does not imply that the position itself is tame.

Example 2.2.2. Returning to Example 2.1.2, consider the game 0.3122, and the genera of the first three heaps:

heap	genus
h_0	0^{120}
h_1	1^{031}
h_2	2^{20}
h_3	0^{02}

Consider h_4. There are two moves from h_4:

$$h_4 \xrightarrow{-3} h_1$$
$$h_4 \xrightarrow{-1} h_3.$$

By Proposition 2.1.2, $\Gamma(h_4) = 2^{1420}$. That is, h_4 is wild, yet all of its options are tame.

Of course, this does not mean that a game with only tame options is automatically wild. Obviously, given a game with only tame options, often this games is also tame, such as h_1, h_2, and h_3 in Example 2.2.2. However, given a game with only tame options, we can classify those into games which will be tame and those which will be wild.

Theorem 2.2.2. *([3], p. 432) Suppose G is a game with only tame options. Then G is wild if and only if amongst the options of G, G has options with genera equal to one, but not both, of 0^{120} or 1^{031}, and options with genera equal to one, but not both, of 0^{02} or 1^{13}.*

Proof. Let $X \subset \mathbb{Z}^{\geq 2}$ such that $|X| < \infty$, and for $n \in X$, there is an option of G with genera $n^{n(n \oplus 2)}$. Let $m = \text{mex}\{0, 1, \{n \mid n \in X\}\}$. Note that $m \geq 2$.

We break the proof into eleven small cases:

1) Suppose G has options with genera equal to 0^{120} and 0^{02}, but no options with genera equal to either 1^{031} or 1^{13}. By Proposition 2.1.2, $\Gamma(G) = 1^{m\cdots}$, which is wild.

2) Suppose G has options with genera equal to 0^{120} and 1^{13}, but no options with genera equal to either 1^{031} or 0^{02}, which are the only tame games with the first superscript equal to zero. By Proposition 2.1.2, $\Gamma(G) = m^{0\cdots}$, which is wild.

3) Suppose G has options with genera equal to 1^{031} and 0^{02}, but no options with genera equal to either 0^{120} or 1^{13}, which are the only tame games with the first superscript equal to one. By Proposition 2.1.2, $\Gamma(G) = m^{1\cdots}$, which is wild.

4) Suppose G has options with genera equal to 1^{031} and 1^{13}, but no options with genera equal to either 0^{120} or 0^{02}. By Proposition 2.1.2, $\Gamma(G) = 0^{m\cdots}$, which is wild.

5) Suppose G has options with genera equal to 0^{120} and 1^{031}. By Proposition 2.1.2, $\Gamma(G) = m^{m(m \oplus 2)}$, which is tame.

6) Suppose G has options with genera equal to 0^{02} and 1^{13}. By Proposition 2.1.2, $\Gamma(G) = m^{m(m \oplus 2)}$, which is tame.

7) Suppose G has no options with genera equal to 0^{120}, 1^{031}, 0^{02}, 1^{13}. That is, all options of G are equal to $n^{n(n\oplus 2)}$ for $n \in X$. By Proposition 2.1.2, $\Gamma(G) = 0^{02}$, which is tame.

8) Suppose that G has an option with genus equal to 0^{120}, and all other options with genera equal to $n^{n(n\oplus 2)}$ for $n \in X$. By the definition of genus, $\Gamma(G) = 1^{g_0 g_1 g_2 \cdots}$. Since the only tame games with first superscript equal to zero are 1^{031} and 0^{02}, neither of which are options of G, $\Gamma(G) = 1^{0 g_1 g_2 \cdots}$. Since the only tame games with second superscript equal to three are 1^{031} and 1^{13}, neither of which are options of G, $0 \oplus 1 = 1$, and an option of G has genus 0^{120}, by Proposition 2.1.2, $\Gamma(G) = 1^{03 g_2 \cdots}$. Since there is an option with genus equal to 0^{120}, $g_3 \geq 1$. Since the only tame games with third superscript equal to one are 1^{031} and 1^{13}, neither of which are options of G and $3 \oplus 1 = 2$, by Proposition 2.1.2, $\Gamma(G) = 1^{031}$, which is tame.

9) Suppose that G has an option with genus equal to 1^{031}, and all other options with genera equal to $n^{n(n\oplus 2)}$ for $n \in X$. By the definition of genus, $\Gamma(G) = 0^{g_0 g_1 g_2 \cdots}$. Since there is an option with genus equal to 1^{031}, $g_0 \geq 1$. Since the only tame games with first superscript equal to one are 1^{13} and 0^{120}, neither of which are options of G, $\Gamma(G) = 0^{1 g_1 g_2 \cdots}$. Since the only tame games with second superscript equal to two are 0^{120} and 0^{02}, neither of which is not an option of G, and $1 \oplus 1 = 0$, by Proposition 2.1.2, $\Gamma(G) = 0^{12 g_2 \cdots}$. Since the only tame games with third superscript equal to zero are 0^{120} and 0^{02}, neither of which are options of G, and $2 \oplus 1 = 3$, by Proposition 2.1.2, $\Gamma(G) = 0^{120}$, which is tame.

10) Suppose that G has an option with genus equal to 0^{02}, and all other options with genera equal to $n^{n(n\oplus 2)}$ for $n \in X$. By the definition of genus, $\Gamma(G) = 1^{g_0 g_1 g_2 \cdots}$. Since G has an option with genus 0^{02}, $g_0 \geq 1$. Since the only tame games with first superscript equal to one are 0^{120} and 1^{13}, neither of which are options of G, by the definition of genus, $\Gamma(G) = 1^{1 g_1 g_2 \cdots}$. Since the only tame games with second superscript equal to three are 1^{031} and 1^{13}, neither of which are options of G, $1 \oplus 1 = 0$, and G has an option with genus equal to 0^{02}, by Proposition 2.1.2, $\Gamma(G) = 1^{13}$, which is tame.

11) Suppose that G has an option with genus equal to 1^{13}, and all other options with

genera equal to $n^{n(n\oplus 2)}$ for $n \in X$. By the definition of genus, $\Gamma(G) = 0^{g_0 g_1 g_2 \cdots}$. Since the only tame games with first superscript equal to zero are 1^{031} and 0^{02}, neither of which are options of G, $\Gamma(G) = 0^{0 g_1 g_2 \cdots}$. Since the only tame games with second superscript equal to two are 0^{02} and 0^{120}, neither of which are options of G and $0 \oplus 1 = 1$, by Proposition 2.1.2, $\Gamma(G) = 0^{02}$, which is tame.

■

2.3 Using the Genus to Determine Outcome Classes

Another important use of genus is that obtain the outcome class of an impartial game G played under the misère game convention through its genus symbol.

Proposition 2.3.1. *([3], p.423) Take an impartial game G. Then under the misère play convention, $G \in \mathcal{P} \iff$ the first superscript in the genus symbol of G equals 0.*

Proof. We will show this by induction on the options of a game.

Suppose G is an impartial game with no options. Then $G \in \mathcal{N}$ and

$$\begin{aligned}\mathcal{G}^-(G) &= 1 \\ &\neq 0.\end{aligned}$$

Suppose now H is an impartial game whose only options are to move to G. Then $H \in \mathcal{P}$ and

$$\begin{aligned}\mathcal{G}^-(H) &= \operatorname{mex}\{\mathcal{G}^-(G)\} \\ &= \operatorname{mex}\{1\} \\ &= 0.\end{aligned}$$

which shows the base case.

Take now an arbitrary game G whose options satisfy the induction hypothesis.

Suppose the first superscript of the genus of G equals 0.

$$\begin{aligned}\mathcal{G}^-(G) &= \operatorname{mex}\{\mathcal{G}^-(G_0), \cdots, \mathcal{G}^-(G_\mu)\} \\ 0 &= \operatorname{mex}\{\mathcal{G}^-(G_0), \cdots, \mathcal{G}^-(G_\mu)\}.\end{aligned}$$

Therefore, $\forall\ i \in \{0, 1, \cdots, \mu\}$, $\mathcal{G}^-(G_i) \neq 0$. By the induction hypothesis, $G_i \in \mathcal{N}$ for each i, therefore $G \in \mathcal{P}$.

Suppose the first superscript of the genus of G does not equal 0. Then there exists an option of G, say G_j such that $\mathcal{G}^-(G_j) = 0$. By the induction hypothesis, $G_j \in \mathcal{P}$, so $G \in \mathcal{N}$. ■

Thus, the genus is the perfect tool to encompass both the normal play and misère play outcome class, since by combining Proposition 1.3.3 and Proposition 2.3.1, we are given the outcome class in both normal and misère play. In the literature, when this is the only information which is sought, the genus of a game is abbreviated to g^{g_0}. However, we will not use this convention as we are often interested in whether a game is tame or not which cannot be determined with only the base and the first superscript of the genus.

2.4 The Algebraic Structure of Tame Games

2.4.1 The Sum of Tame Games

Before we begin, we recall the following result from [1]:

Lemma 2.4.1. *For games G, and H, $\mathcal{G}^+(G+H) = \mathcal{G}^+(G) \oplus \mathcal{G}^+(H)$. ([1], p.114)*

Theorem 2.4.2. *([4], p.145) Let G and H be two tame games. Then $G+H$ is a tame game also with:*

$$\Gamma\left(G+H\right) = \begin{cases} \Gamma\left(H\right) & \textit{if}\ \Gamma\left(G\right) = 0^{120} \\ 0^{120} & \textit{if}\ \Gamma\left(G\right) = \Gamma\left(H\right) = 1^{031} \\ (n \oplus 1)^{(n\oplus 1)(n\oplus 3)} & \textit{if}\ \Gamma\left(G\right) = 1^{031},\ \Gamma\left(H\right) = n^{n(n\oplus 2)} \\ (n \oplus m)^{(n\oplus m)(n\oplus m\oplus 2)} & \textit{if}\ \Gamma\left(G\right) = n^{n(n\oplus 2)},\ \Gamma\left(H\right) = m^{m(m\oplus 2)}. \end{cases}$$

Proof. We proceed by induction on the options of the sum of G and H.

Define the following games:

- G_0 is a game which has no options. Then $\Gamma\left(G_0\right) = 0^{120}$.
- G_1 is a game whose only option is to G_0. Then $\Gamma\left(G_1\right) = 1^{031}$.

- G_2 is a game whose only options are to G_0 and G_1. Then $\Gamma(G_2) = 2^{20}$.

Since G_0 has no options, for any game H, the options of G_0+H are the same as the options of H. Since the genus of a game depends only on the options, $\Gamma(G_0 + H) = \Gamma(H)$.

Suppose we have the sum of games G_1 and G_1. Then

$$
\begin{aligned}
\mathcal{G}^+(G_1 + G_1) &= \mathcal{G}^+(G_1) \oplus \mathcal{G}^+(G_1) \text{ by Lemma 2.4.1} \\
&= 1 \oplus 1 \\
&= 0, \\
\mathcal{G}^-(G_1 + G_1) &= \operatorname{mex}\{\mathcal{G}^-(G_1 + G_0), \mathcal{G}^-(G_0 + G_1)\} \\
&= \operatorname{mex}\{\mathcal{G}^-(G_1)\} \\
&= \operatorname{mex}\{0\} \\
&= 1, \\
\mathcal{G}^-(G_1 + G_1 + \mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^-(G_1 + G_1), \mathcal{G}^-(G_1 + G_1 + \mathbb{1}), \mathcal{G}^-(G_1 + G_0 + \mathbb{2}), \\
&\qquad \mathcal{G}^-(G_0 + G_1 + \mathbb{2})\} \\
&= \operatorname{mex}\{\ \mathcal{G}^-(G_1 + G_1), \mathcal{G}^-(G_1 + G_1) \oplus 1, \mathcal{G}^-(G_1 + G_0 + \mathbb{2}), \\
&\qquad \mathcal{G}^-(G_0 + G_1 + \mathbb{2})\} \text{ by Lemma 2.1.1} \\
&= \operatorname{mex}\{1, 0, \mathcal{G}^-(G_1 + \mathbb{2})\} \\
&= \operatorname{mex}\{1, 0, 3\} \\
&= 2, \\
\mathcal{G}^-(G_1 + G_1 + \mathbb{2} + \mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^-(G_1 + G_1 + \mathbb{2}), \mathcal{G}^-(G_1 + G_1 + \mathbb{2} + \mathbb{1}), \\
&\qquad \mathcal{G}^-(G_1 + G_0 + \mathbb{2} + \mathbb{2}), \mathcal{G}^-(G_0 + G_1 + \mathbb{2} + \mathbb{2})\} \\
&= \operatorname{mex}\{\mathcal{G}^-(G_1 + G_1 + \mathbb{2}), \mathcal{G}^-(G_1 + G_1 + \mathbb{2}) \oplus 1, \\
&\qquad \mathcal{G}^-(G_1 + G_0 + \mathbb{2} + \mathbb{2}), \mathcal{G}^-(G_0 + G_1 + \mathbb{2} + \mathbb{2})\} \\
&\qquad \text{by Lemma 2.1.1} \\
&= \operatorname{mex}\{2, 3, \mathcal{G}^-(G_1 + \mathbb{2} + \mathbb{2})\} \\
&= \operatorname{mex}\{2, 3, 1\} \\
&= 0.
\end{aligned}
$$

Therefore $\Gamma\left(G_1 + G_1\right) = 0^{120}$.

Suppose we have the sum of games G_1 and G_2. Then

$$
\begin{aligned}
\mathcal{G}^{+}(G_1 + G_2) &= \mathcal{G}^{+}(G_1) \oplus \mathcal{G}^{+}(G_2) \text{ by Lemma 2.4.1} \\
&= 1 \oplus 2 \\
&= 3, \\
\mathcal{G}^{-}(G_1 + G_2) &= \operatorname{mex}\{\mathcal{G}^{-}(G_0 + G_2), \mathcal{G}^{-}(G_1 + G_0), \mathcal{G}^{-}(G_1 + G_1)\} \\
&= \operatorname{mex}\{\mathcal{G}^{-}(G_2), \mathcal{G}^{-}(G_1), 1\} \\
&= \operatorname{mex}\{2, 0, 1\} \\
&= 3, \\
\mathcal{G}^{-}(G_1 + G_2 + \mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^{-}(G_1 + G_2), \mathcal{G}^{-}(G_1 + G_2 + \mathbb{1}), \mathcal{G}^{-}(G_0 + G_2 + \mathbb{2}), \\
&\quad\ \mathcal{G}^{-}(G_1 + G_0 + \mathbb{2}), \mathcal{G}^{-}(G_1 + G_1 + \mathbb{2})\} \\
&= \operatorname{mex}\{\mathcal{G}^{-}(G_1 + G_2), \mathcal{G}^{-}(G_1 + G_2) \oplus 1, \mathcal{G}^{-}(G_0 + G_2 + \mathbb{2}), \\
&\quad\ \mathcal{G}^{-}(G_1 + G_0 + \mathbb{2}), \mathcal{G}^{-}(G_1 + G_1 + \mathbb{2})\} \text{ by Lemma 2.1.1} \\
&= \operatorname{mex}\{3, 2, \mathcal{G}^{-}(G_2 + \mathbb{2}), \mathcal{G}^{-}(G_1 + \mathbb{2}), 2\} \\
&= \operatorname{mex}\{3, 2, 0, 3, 2\} \\
&= 1, \\
\mathcal{G}^{-}(G_1 + G_2 + \mathbb{2} + \mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^{-}(G_1 + G_2 + \mathbb{2}), \mathcal{G}^{-}(G_1 + G_2 + \mathbb{2} + \mathbb{1}), \\
&\quad\ \mathcal{G}^{-}(G_0 + G_2 + \mathbb{2} + \mathbb{2}), \mathcal{G}^{-}(G_1 + G_0 + \mathbb{2} + \mathbb{2}), \\
&\quad\ \mathcal{G}^{-}(G_1 + G_1 + \mathbb{2} + \mathbb{2})\} \\
&= \operatorname{mex}\{\mathcal{G}^{-}(G_1 + G_2 + \mathbb{2}), \mathcal{G}^{-}(G_1 + G_2 + \mathbb{2}) \oplus 1, \\
&\quad\ \mathcal{G}^{-}(G_0 + G_2 + \mathbb{2} + \mathbb{2}), \mathcal{G}^{-}(G_1 + G_0 + \mathbb{2} + \mathbb{2}), \\
&\quad\ \mathcal{G}^{-}(G_1 + G_1 + \mathbb{2} + \mathbb{2})\} \text{ by Lemma 2.1.1} \\
&= \operatorname{mex}\{1, 0, \mathcal{G}^{-}(G_2 + \mathbb{2} + \mathbb{2}), \mathcal{G}^{-}(G_1 + \mathbb{2} + \mathbb{2}), 0\} \\
&= \operatorname{mex}\{1, 0, 2, 1, 0\} \\
&= 3.
\end{aligned}
$$

Therefore $\Gamma\left(G_1 + G_2\right) = 3^{31} = (1 \oplus 2)^{(1 \oplus 2)((1 \oplus 2) \oplus 2)}$.

Suppose we have the sum of games G_2 and G_2. Then

$$\begin{aligned}
\mathcal{G}^{+}(G_2+G_2) &= \mathcal{G}^{+}(G_2)\oplus\mathcal{G}^{+}(G_2) \text{ by Lemma 2.4.1}\\
&= 2\oplus 2\\
&= 0,\\
\mathcal{G}^{-}(G_2+G_2) &= \operatorname{mex}\{\mathcal{G}^{-}(G_0+G_2),\mathcal{G}^{-}(G_1+G_2)\}\\
&= \operatorname{mex}\{\mathcal{G}^{-}(G_2),3\}\\
&= \operatorname{mex}\{2,3\}\\
&= 0,\\
\mathcal{G}^{-}(G_2+G_2+\mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^{-}(G_2+G_2),\mathcal{G}^{-}(G_2+G_2+\mathbb{1}),\mathcal{G}^{-}(G_0+G_2+\mathbb{2}),\\
&\quad\ \mathcal{G}^{-}(G_1+G_2+\mathbb{2})\}\\
&= \operatorname{mex}\{\mathcal{G}^{-}(G_2+G_2),\mathcal{G}^{-}(G_2+G_2)\oplus 1,\mathcal{G}^{-}(G_0+G_2+\mathbb{2}),\\
&\quad\ \mathcal{G}^{-}(G_1+G_2+\mathbb{2})\} \text{ by Lemma 2.1.1}\\
&= \operatorname{mex}\{0,1,\mathcal{G}^{-}(G_2+\mathbb{2}),1\}\\
&= \operatorname{mex}\{0,1,0,1\}\\
&= 2,\\
\mathcal{G}^{-}(G_2+G_2+\mathbb{2}+\mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^{-}(G_2+G_2+\mathbb{2}),\mathcal{G}^{-}(G_2+G_2+\mathbb{2}+\mathbb{1}),\\
&\quad\ \mathcal{G}^{-}(G_0+G_2+\mathbb{2}+\mathbb{2}),\mathcal{G}^{-}(G_1+G_2+\mathbb{2}+\mathbb{2})\}\\
&= \operatorname{mex}\{\mathcal{G}^{-}(G_2+G_2+\mathbb{2}),\mathcal{G}^{-}(G_2+G_2+\mathbb{2})\oplus 1,\\
&\quad\ \mathcal{G}^{-}(G_0+G_2+\mathbb{2}+\mathbb{2}),\mathcal{G}^{-}(G_1+G_2+\mathbb{2}+\mathbb{2})\}\\
&\quad\ \text{by Lemma 2.1.1}\\
&= \operatorname{mex}\{2,3,\mathcal{G}^{-}(G_2+\mathbb{2}+\mathbb{2}),3\}\\
&= \operatorname{mex}\{2,3,2,3\}\\
&= 0.
\end{aligned}$$

Therefore $\Gamma\left(G_2+G_2\right)=0^{02}=(2\oplus 2)^{(2\oplus 2)((2\oplus 2)\oplus 2)}$.

This completes the base case.

Suppose now we have games G, and H, such that the induction hypothesis holds for $G+H$.

We break our proof into the following three cases: $\Gamma(G) = 0^{120}$, $\Gamma(G) = 1^{031}$, and $\Gamma(G) = n^{n(n\oplus 2)}$ for $n \in \mathbb{Z}^{\geq 0}$.

1) Suppose now we have games G and H, with $\Gamma(G) = 0^{120}$. Claim $\Gamma(G+H) = \Gamma(H)$.

   We further decompose our proof into three cases: $\Gamma(H) = 0^{120}$, $\Gamma(H) = 1^{031}$, and $\Gamma(H) = n^{n(n\oplus 2)}$ for $n \in \mathbb{Z}^{\geq 0}$.

   a) Suppose we have games G and H, both having genus 0^{120}. Claim that $G+H$ has genus 0^{120}.

$$\begin{aligned}
\mathcal{G}^+(G+H) &= \mathcal{G}^+(G) \oplus \mathcal{G}^+(H) \text{ by Lemma 2.4.1} \\
&= 0 \oplus 0 \\
&= 0, \\
\mathcal{G}^-(G+H) &= \text{mex}\{\{\mathcal{G}^-(G'+H) \mid G' \text{ an option of } G\}, \\
&\quad\ \{\mathcal{G}^-(G+H') \mid H' \text{ an option of } H\}\} \\
&= \text{mex}\{\{\mathcal{G}^-(G') \mid G' \text{ an option of } G\}, \\
&\quad\ \{\mathcal{G}^-(H') \mid H' \text{ an option of } H\}\} \text{ by induction.}
\end{aligned}$$

   Since $\mathcal{G}^-(G) = 1$, there is some option of G, say G', such that $\mathcal{G}^-(G') = 0$ and all other options of G have $\mathcal{G}^-$ value either zero or value greater than one. The same holds for H. Therefore $\mathcal{G}^-(G+H) = 1$.

$$\begin{aligned}
\mathcal{G}^-(G+H+2) &= \text{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H+\mathbb{1}) \\
&\quad\ \{\mathcal{G}^-(G'+H+2) \mid G' \text{ an option of } G\}, \\
&\quad\ \{\mathcal{G}^-(G+H'+2) \mid H' \text{ an option of } H\}\} \\
&= \text{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H) \oplus 1, \\
&\quad\ \{\mathcal{G}^-((G'+2)+H) \mid G' \text{ an option of } G\}, \\
&\quad\ \{\mathcal{G}^-((H'+2)+G) \mid H' \text{ an option of } H\}\} \\
&\quad\ \text{by Lemma 2.1.1} \\
&= \text{mex}\{1, 0, \{\mathcal{G}^-((G'+2)+H) \mid G' \text{ an option of } G\},
\end{aligned}$$

$$
\begin{aligned}
& \{\mathcal{G}^-((H'+\mathbb{2})+G) \mid H' \text{ an option of } H\}\} \\
= \;& \text{mex}\{0,1,\{\mathcal{G}^-(G'+\mathbb{2}) \mid G' \text{ an option of } G\}, \\
& \{\mathcal{G}^-(H'+\mathbb{2}) \mid H' \text{ an option of } H\}\} \text{ by induction.}
\end{aligned}
$$

Suppose there is some G' such that $\mathcal{G}^-(G'+\mathbb{2}) = 2$. Then $\mathcal{G}^-(G+\mathbb{2}) \neq 2$, which is a contradiction. Similarly for H. Therefore $\mathcal{G}^-(G+H+\mathbb{2}) = 2$.

$$
\begin{aligned}
\mathcal{G}^-(G+H+\mathbb{2}+\mathbb{2}) &= \text{mex}\{\mathcal{G}^-(G+H+\mathbb{2}), \mathcal{G}^-(G+H+\mathbb{2}+\mathbb{1}), \\
&\quad \{\mathcal{G}^-(G'+H+\mathbb{2}+\mathbb{2}) \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-(G+H'+\mathbb{2}+\mathbb{2}) \mid H' \text{ an option of } H\}\} \\
&= \text{mex}\{\mathcal{G}^-(G+H+\mathbb{2}), \mathcal{G}^-(G+H+\mathbb{2}) \oplus 1, \\
&\quad \{\mathcal{G}^-((G'+\mathbb{2}+\mathbb{2})+H) \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-((H'+\mathbb{2}+\mathbb{2})+G) \mid H' \text{ an option of } H\}\} \\
&\quad \text{by Lemma 2.1.1} \\
&= \text{mex}\{2,3, \\
&\quad \{\mathcal{G}^-((G'+\mathbb{2}+\mathbb{2})+H) \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-((H'+\mathbb{2}+\mathbb{2})+G) \mid H' \text{ an option of } H\}\} \\
&= \text{mex}\{2,3,\{\mathcal{G}^-(G'+\mathbb{2}+\mathbb{2}) \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-(H'+\mathbb{2}+\mathbb{2}) \mid H' \text{ an option of } H\}\} \\
&\quad \text{by induction.}
\end{aligned}
$$

Suppose there is some G' such that $\mathcal{G}^-(G'+\mathbb{2}+\mathbb{2}) = 0$. Then $\mathcal{G}^-(G+\mathbb{2}+\mathbb{2}) \neq 0$, which is a contradiction. Therefore $\mathcal{G}^-(G+H+\mathbb{2}+\mathbb{2}) = 0$.

Therefore, the genus of $G+H$ is 0^{120}.

b) Suppose now that H has genus 1^{031}. Claim that $G+H$ has genus 1^{031}.

$$
\begin{aligned}
\mathcal{G}^+(G+H) &= \mathcal{G}^+(G) \oplus \mathcal{G}^+(H) \text{ by Lemma 2.4.1} \\
&= 0 \oplus 1 \\
&= 1, \\
\mathcal{G}^-(G+H) &= \text{mex}\{\{\mathcal{G}^-(G'+H) \mid G' \text{ an option of } G\},
\end{aligned}
$$

$$
\begin{aligned}
& \qquad \{\mathcal{G}^-(G+H') \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{\{\mathcal{G}^-(G') \oplus 1 \mid G' \text{ an option of } G\}, \\
& \qquad \{\mathcal{G}^-(H') \mid H' \text{ an option of } H\}\} \text{ by induction.}
\end{aligned}
$$

For every option H', $\mathcal{G}^-(H') \geq 1$ since $\mathcal{G}^-(H) = 0$.

Suppose there exists option of G, G' such that $\mathcal{G}^-(G') \oplus 1 = 0$. Then $\mathcal{G}^-(G') = 1$, which contradicts $\mathcal{G}^-(G) = 1$. Therefore $\mathcal{G}^-(G+H) = 0$.

$$
\begin{aligned}
\mathcal{G}^-(G+H+\mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H+\mathbb{1}), \\
& \qquad \{\mathcal{G}^-(G'+H+\mathbb{2}) \mid G' \text{ an option of } G\}, \\
& \qquad \{\mathcal{G}^-(G+H'+\mathbb{2}) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H) \oplus 1, \\
& \qquad \{\mathcal{G}^-((G'+\mathbb{2})+H) \mid G' \text{ an option of } G\}, \\
& \qquad \{\mathcal{G}^-((H'+\mathbb{2})+G) \mid H' \text{ an option of } H\}\} \\
& \qquad \text{by Lemma 2.1.1} \\
&= \operatorname{mex}\{0, 1, \{\mathcal{G}^-((G'+\mathbb{2})+H) \mid G' \text{ an option of } G\}, \\
& \qquad \{\mathcal{G}^-((H'+\mathbb{2})+G) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{0, 1, \{\mathcal{G}^-(G'+\mathbb{2}) \oplus 1 \mid G' \text{ an option of } G\}, \\
& \qquad \{\mathcal{G}^-(H'+\mathbb{2}) \mid H' \text{ an option of } H\}\} \text{ by induction.}
\end{aligned}
$$

Since $\mathcal{G}^-(H+\mathbb{2}) = 3$ and $\mathcal{G}^-(H) = 0$, there exists an option of H, H', such that $\mathcal{G}^-(H') = 2$, and no option H' such that $\mathcal{G}^-(H'+\mathbb{2}) = 3$.

Suppose there exists option of G, G', such that $\mathcal{G}^-(G'+\mathbb{2}) \oplus 1 = 3$. Then $\mathcal{G}^-(G'+\mathbb{2}) = 2$, which contradicts $\mathcal{G}^-(G+\mathbb{2}) = 2$.

Therefore $\mathcal{G}^-(G+H+\mathbb{2}) = 3$.

$$
\begin{aligned}
\mathcal{G}^-(G+H+\mathbb{2}+\mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^-(G+H+\mathbb{2}), \mathcal{G}^-(G+H+\mathbb{2}+\mathbb{1}), \\
& \qquad \{\mathcal{G}^-(G'+H+\mathbb{2}+\mathbb{2}) \mid G' \text{ an option of } G\}, \\
& \qquad \{\mathcal{G}^-(G+H'+\mathbb{2}+\mathbb{2}) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{\mathcal{G}^-(G+H+\mathbb{2}), \mathcal{G}^-(G+H+\mathbb{2}) \oplus 1, \\
& \qquad \{\mathcal{G}^-((G'+\mathbb{2}+\mathbb{2})+H) \mid G' \text{ an option of } G\},
\end{aligned}
$$

$$
\begin{aligned}
&\quad \{\mathcal{G}^-((H'+2+2)+G) \mid H' \text{ an option of } H\}\} \\
&\quad \text{by Lemma 2.1.1} \\
&= \text{mex}\{3, 2, \\
&\quad \{\mathcal{G}^-((G'+2+2)+H) \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-((H'+2+2)+G) \mid H' \text{ an option of } H\}\} \\
&= \text{mex}\{3, 2, \\
&\quad \{\mathcal{G}^-(G'+2+2) \oplus 1 \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-(H'+2+2) \mid H' \text{ an option of } H\}\} \\
&\quad \text{by induction.}
\end{aligned}
$$

Since $\mathcal{G}^-(H+2+2) = 1$ and $\mathcal{G}^-(H+2) = 3$, there exists an option of H, H', such that $\mathcal{G}^-(H'+2) = 0$, and no option of H such that $\mathcal{G}^-(H'+2+2) = 1$. Suppose there exists option of G, G', such that $\mathcal{G}^-(G'+2+2) \oplus 1 = 1$. Then $\mathcal{G}^-(G'+2+2) = 0$, which contradicts $\mathcal{G}^-(G+2+2) = 0$.

Therefore $\mathcal{G}^-(G+H+2+2) = 1$.

Thus, $\Gamma(G+H) = 1^{031}$.

c) Suppose now that H has genus $n^{n(n\oplus 2)}$ for $n \in \mathbb{Z}^{\geq 0}$. Claim that $G+H$ has genus $n^{n(n\oplus 2)}$.

$$
\begin{aligned}
\mathcal{G}^+(G+H) &= \mathcal{G}^+(G) \oplus \mathcal{G}^+(H) \text{ by Lemma 2.4.1} \\
&= 0 \oplus n \\
&= n, \\
\mathcal{G}^-(G+H) &= \text{mex}\{\{\mathcal{G}^-(G'+H) \mid G' \text{ an option of } G\}, \\
&\qquad \{\mathcal{G}^-(G+H') \mid H' \text{ an option of } H\}\} \\
&= \text{mex}\{\{\mathcal{G}^-(G'+H) \mid G' \text{ an option of } G\}, \\
&\qquad \{\mathcal{G}^-(H') \mid H' \text{ an option of } H\}\} \text{ by induction.}
\end{aligned}
$$

By induction, we have

$$
\mathcal{G}^-(G'+H) = \begin{cases} \mathcal{G}^-(G') \oplus n & \text{if } \Gamma(G') \neq 1^{031} \\ 1 \oplus n & \text{if } \Gamma(G') = 1^{031}. \end{cases}
$$

Consider when $\Gamma(G') \neq 1^{031}$. Suppose that $\mathcal{G}^-(G') \oplus n = n$. Then $\mathcal{G}^-(G') = 0$, and since G' is tame, $\Gamma(G') = 0^{02}$ or 1^{031}. If $\Gamma(G') = 0^{02}$, then $\Gamma(G) \neq 0^{120}$, and we took $\Gamma(G') \neq 1^{031}$.

Consider now when $\Gamma(G') = 1^{031}$. We see that $1 \oplus n \neq n$.

Since $\mathcal{G}^-(H) = n$, $\forall\ u < n$, there exists option H' of H such that $\mathcal{G}^-(H') = u$.

Therefore $\mathcal{G}^-(G+H) = n$.

$$\begin{aligned}
\mathcal{G}^-(G+H+\mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H+\mathbb{1}) \\
&\quad \{\mathcal{G}^-(G'+H+\mathbb{2}) \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-(G+H'+\mathbb{2}) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H) \oplus 1, \\
&\quad \{\mathcal{G}^-((G'+\mathbb{2})+H) \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-((H'+\mathbb{2})+G) \mid H' \text{ an option of } H\}\} \\
&\quad \text{by Lemma 2.1.1} \\
&= \operatorname{mex}\{n, n \oplus 1, \{\mathcal{G}^-((G'+\mathbb{2})+H) \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-((H'+\mathbb{2})+G) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{n, n \oplus 1, \{\mathcal{G}^-((G'+\mathbb{2})+H \mid G' \text{ an option of } G\}, \\
&\quad \{\mathcal{G}^-(H'+\mathbb{2}) \mid H' \text{ an option of } H\}\} \text{ by induction.}
\end{aligned}$$

By induction, we have

$$\mathcal{G}^-(G+H+\mathbb{2}) = \begin{cases} \mathcal{G}^-(G') \oplus n \oplus 2 & \text{if } \Gamma(G') \neq 1^{031} \\ 1 \oplus n \oplus 2 & \text{if } \Gamma(G') = 1^{031}. \end{cases}$$

Consider when $\Gamma(G') \neq 1^{031}$. Suppose that $\mathcal{G}^-(G') \oplus n \oplus 2 = n \oplus 2$. Then $\mathcal{G}^-(G') = 0$, and since G' is tame, $\Gamma(G') = 0^{02}$ or 1^{031}. If $\Gamma(G') = 0^{02}$, then $\Gamma(G) \neq 0^{120}$, and we took $\Gamma(G') \neq 1^{031}$.

Consider now when $\Gamma(G') = 1^{031}$. We see that $1 \oplus n \oplus 2 \neq n \oplus 2$.

Since $\mathcal{G}^-(H+\mathbb{2}) = n \oplus 2$, $\forall\ u < n$, there exists option H' of H such that $\mathcal{G}^-(H'+\mathbb{2}) = u$.

Therefore $\mathcal{G}^-(G+H+\mathbb{2}) = n \oplus 2$.

As per Corollary 2.1.6, we must also show that $\mathcal{G}^-(G+H+2+2)=n$. The argument to show this is similar to the one given above.

2) Suppose we have games G and H with $\Gamma(G)=1^{031}$, and $\Gamma(H)\neq 0^{120}$. Claim

$$\Gamma(G+H)=\begin{cases} 0^{120} & \text{if } \Gamma(G)=\Gamma(H)=1^{031} \\ (n\oplus 1)^{(n\oplus 1)(n\oplus 3)} & \text{if } \Gamma(G)=1^{031},\ \Gamma(H)=n^{n(n\oplus 2)}. \end{cases}$$

a) Suppose that $\Gamma(H)=1^{031}$. Claim that $\Gamma(G+H)=0^{120}$.

$$\begin{aligned} \mathcal{G}^+(G+H) &= \mathcal{G}^+(G)\oplus\mathcal{G}^+(H) \text{ by Lemma 2.4.1} \\ &= 1\oplus 1 \\ &= 0, \\ \mathcal{G}^-(G+H) &= \text{mex}\{\{\mathcal{G}^-(G'+H)\mid G' \text{ an option of } G\}, \\ &\quad \{\mathcal{G}^-(G+H')\mid H' \text{ an option of } H\}\} \\ &= \text{mex}\{\{\mathcal{G}^-(G')\oplus 1\mid G' \text{ an option of } G\}, \\ &\quad \{\mathcal{G}^-(H')\oplus 1\mid H' \text{ an option of } H\}\} \text{ by induction.} \end{aligned}$$

Suppose there exists an option of G, G' with $\mathcal{G}^-(G')\oplus 1=1$. Then $\mathcal{G}^-(G')=0$, which contradicts $\mathcal{G}^-(G)=0$. Similarly, there does not exist an option of H, H' with $\mathcal{G}^-(H')=1$.

Claim: since $\Gamma(G)=1^{031}$ and G tame, there is an option G' with $\Gamma(G')=0^{120}$. Since $\mathcal{G}^+(G)=1$, there exists option G' with $\mathcal{G}^+(G')=0$. Since G' is tame, $\Gamma(G')=0^{02}$ or 0^{120}. Since $\mathcal{G}^-(G)=0$, $\mathcal{G}^-(G')\neq 0$, so $\Gamma(G')=0^{120}$.

Thus, there exists an option of G, G' with $\mathcal{G}^-(G')=1$. Then $\mathcal{G}^-(G')\oplus 1=0$, so $\mathcal{G}^-(G+H)=1$.

$$\begin{aligned} \mathcal{G}^-(G+H+2) &= \text{mex}\{\mathcal{G}^-(G+H),\mathcal{G}^-(G+H+1) \\ &\quad \{\mathcal{G}^-(G'+H+2)\mid G' \text{ an option of } G\}, \\ &\quad \{\mathcal{G}^-(G+H'+2)\mid H' \text{ an option of } H\}\} \\ &= \text{mex}\{\mathcal{G}^-(G+H),\mathcal{G}^-(G+H)\oplus 1, \\ &\quad \{\mathcal{G}^-((G'+2)+H)\mid G' \text{ an option of } G\}, \\ &\quad \{\mathcal{G}^-((H'+2)+G)\mid H' \text{ an option of } H\}\} \end{aligned}$$

$$\begin{aligned}
& \quad \text{by Lemma 2.1.1} \\
&= \operatorname{mex}\{1, 0, \{\mathcal{G}^-((G' + \mathbb{2}) + H) \mid G' \text{ an option of } G\}, \\
& \quad \{\mathcal{G}^-((H' + \mathbb{2}) + G) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{0, 1, \{\mathcal{G}^-(G' + \mathbb{2}) \oplus 1 \mid G' \text{ an option of } G\}, \\
& \quad \{\mathcal{G}^-(H' + \mathbb{2}) \oplus 1 \mid H' \text{ an option of } H\}\} \\
& \quad \text{by induction.}
\end{aligned}$$

Suppose there is some G' such that $\mathcal{G}^-(G' + \mathbb{2}) \oplus 1 = 2$. Then $\mathcal{G}^-(G' + \mathbb{2}) = 3$, which contradicts $\mathcal{G}^-(G + \mathbb{2}) = 3$. Similarly there does not exist an option of H, H' with $\mathcal{G}^-(H' + \mathbb{2}) \oplus 1 = 2$. Therefore $\mathcal{G}^-(G + H + \mathbb{2}) = 2$.

$$\begin{aligned}
\mathcal{G}^-(G + H + \mathbb{2} + \mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^-(G + H + \mathbb{2}), \mathcal{G}^-(G + H + \mathbb{2} + \mathbb{1}), \\
& \quad \{\mathcal{G}^-(G' + H + \mathbb{2} + \mathbb{2}) \mid G' \text{ an option of } G\}, \\
& \quad \{\mathcal{G}^-(G + H' + \mathbb{2} + \mathbb{2}) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{\mathcal{G}^-(G + H + \mathbb{2}), \mathcal{G}^-(G + H + \mathbb{2}) \oplus 1, \\
& \quad \{\mathcal{G}^-((G' + \mathbb{2} + \mathbb{2}) + H) \mid G' \text{ an option of } G\}, \\
& \quad \{\mathcal{G}^-((H' + \mathbb{2} + \mathbb{2}) + G) \mid H' \text{ an option of } H\}\} \\
& \quad \text{by Lemma 2.1.1} \\
&= \operatorname{mex}\{2, 3, \\
& \quad \{\mathcal{G}^-((G' + \mathbb{2} + \mathbb{2}) + H) \mid G' \text{ an option of } G\}, \\
& \quad \{\mathcal{G}^-((H' + \mathbb{2} + \mathbb{2}) + G) \mid H' \text{ an option of } H\}\} \\
& \quad \text{by Lemma 2.1.1} \\
&= \operatorname{mex}\{2, 3, \\
& \quad \{\mathcal{G}^-(G' + \mathbb{2} + \mathbb{2}) \oplus 1 \mid G' \text{ an option of } G\}, \\
& \quad \{\mathcal{G}^-(H' + \mathbb{2} + \mathbb{2}) \oplus 1 \mid H' \text{ an option of } H\}\} \\
& \quad \text{by induction.}
\end{aligned}$$

Suppose there is some G' such that $\mathcal{G}^-(G' + \mathbb{2} + \mathbb{2}) \oplus 1 = 0$. Then $\mathcal{G}^-(G' + \mathbb{2} + \mathbb{2}) = 1$, which contradicts $\mathcal{G}^-(G + \mathbb{2} + \mathbb{2}) = 1$. Similarly there does not exist an option of H, H' with $\mathcal{G}^-(H' + \mathbb{2} + \mathbb{2}) \oplus 1 = 0$. Thus $\mathcal{G}^-(G + H + \mathbb{2} + \mathbb{2}) = 0$.

Therefore, $\Gamma(G+H) = 0^{120}$.

b) Suppose that $\Gamma(H) = n^{n(n\oplus 2)}$. Claim that $\Gamma(G+H) = (n\oplus 1)^{(n\oplus 1)(n\oplus 3)}$.

$$\begin{aligned}
\mathcal{G}^+(G+H) &= \mathcal{G}^+(G) \oplus \mathcal{G}^+(H) \text{ by Lemma 2.4.1}\\
&= 1 \oplus n,\\
\mathcal{G}^-(G+H) &= \text{mex}\{\{\mathcal{G}^-(G'+H) \mid G' \text{ an option of } G\},\\
&\quad \{\mathcal{G}^-(G+H') \mid H' \text{ an option of } H\}\}\\
&= \text{mex}\{\{\mathcal{G}^-(G'+H) \mid G' \text{ an option of } G\},\\
&\quad \{\mathcal{G}^-(H') \oplus 1 \mid H' \text{ an option of } H\}\} \text{ by induction.}
\end{aligned}$$

By induction, we have

$$\mathcal{G}^-(G'+H) = \begin{cases} \mathcal{G}^-(G') \oplus n & \text{if } \Gamma(G') \neq 0^{120} \\ n & \text{if } \Gamma(G') = 0^{120}. \end{cases}$$

Consider when $\Gamma(G') \neq 0^{120}$. Suppose that $\mathcal{G}^-(G') \oplus n = n \oplus 1$. Then $\mathcal{G}^-(G') = 1$. Since G' is tame, $\Gamma(G) = 1^{13}$ or 0^{120}. If $\Gamma(G') = 1^{13}$, then $\Gamma(G) \neq 1^{031}$, and we took $\Gamma(G') \neq 0^{120}$

Consider now when $\Gamma(G') = 1^{031}$ We see that $n \neq n \oplus 1$.

Suppose there exists option of H, H', such that $\mathcal{G}^-(H') \oplus 1 = n \oplus 1$. Then $\mathcal{G}^-(H') = n$, which contradicts $\mathcal{G}^-(H) = n$.

Since $\mathcal{G}^-(H) = n$, $\forall\, u < n$, there exists option H' of H such that $\mathcal{G}^-(H') = u$. Hence, $\forall\, j < n \oplus 1$, there exists H' such that $\mathcal{G}^-(H') \oplus 1 = j$.

Therefore $\mathcal{G}^-(G+H) = n \oplus 1$.

$$\begin{aligned}
\mathcal{G}^-(G+H+2) &= \text{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H+1)\\
&\quad \{\mathcal{G}^-(G'+H+2) \mid G' \text{ an option of } G\},\\
&\quad \{\mathcal{G}^-(G+H'+2) \mid H' \text{ an option of } H\}\}\\
&= \text{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H) \oplus 1,\\
&\quad \{\mathcal{G}^-((G'+2)+H) \mid G' \text{ an option of } G\},\\
&\quad \{\mathcal{G}^-((H'+2)+G) \mid H' \text{ an option of } H\}\}\\
&\quad \text{by Lemma 2.1.1}
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{mex}\{n \oplus 1, n, \\
&\qquad \{\mathcal{G}^-((G' + 2) + H) \mid G' \text{ an option of } G\}, \\
&\qquad \{\mathcal{G}^-((H' + 2) + G) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{n \oplus 1, n, \\
&\qquad \{\mathcal{G}^-((G' + 2) + H) \mid G' \text{ an option of } G\}, \\
&\qquad \{\mathcal{G}^-(H' + 2) \oplus 1 \mid H' \text{ an option of } H\}\} \\
&\qquad \text{by induction.}
\end{aligned}
$$

By induction, we have

$$
\mathcal{G}^-((G' + 2) + H) = \begin{cases} \mathcal{G}^-(G') \oplus n \oplus 2 & \text{if } \Gamma(G') \neq 0^{120} \\ n \oplus 2 & \text{if } \Gamma(G') = 0^{120}. \end{cases}
$$

Consider when $\Gamma(G') \neq 0^{120}$. Suppose that $\mathcal{G}^-(G') \oplus n \oplus 2 = n \oplus 3$. Then $\mathcal{G}^-(G') = 1$. Since G' is tame, $\Gamma(G) = 1^{13}$ or 0^{120}. If $\Gamma(G') = 1^{13}$, then $\Gamma(G) \neq 1^{031}$, and we took $\Gamma(G') \neq 0^{120}$

Consider now when $\Gamma(G') = 1^{031}$. We see that $n \oplus 2 \neq n \oplus 3$.

Recall that since $\Gamma(G) = 1^{031}$, there is an option G' with $\Gamma(G') = 0^{120}$. Thus, one of the values in the mex set is $n \oplus 2$.

Suppose there exists option of H, H', such that $\mathcal{G}^-(H' + 2) \oplus 1 = n \oplus 3$. Then $\mathcal{G}^-(H' + 2) = n \oplus 2$, which contradicts $\mathcal{G}^-(H) = n \oplus 2$.

Since $\mathcal{G}^-(H + 2) = n \oplus 2$, $\forall\ u < n$, there exists option H' of H such that $\mathcal{G}^-(H' + 2) = u$. Hence, $\forall\, j \leq n \oplus 1$, there exists H' such that $\mathcal{G}^-(H' + 2) \oplus 1 = j$.

Therefore $\mathcal{G}^-(G + H + 2) = n \oplus 3$.

As per Corollary 2.1.6, we must also show that $\mathcal{G}^-(G + H + 2 + 2) = n \oplus 1$. The argument to show this is similar to the one given above.

Therefore $\Gamma(G + H) = (n \oplus 1)^{(n \oplus 1)(n \oplus 3)}$.

3) Suppose we have games G and H with $\Gamma(G) = n^{n(n \oplus 2)}$, $\Gamma(H) = m^{m(m \oplus 2)}$. Claim that $\Gamma(G + H) = (n \oplus m)^{(n \oplus m)(n \oplus m \oplus 2)}$.

$$
\mathcal{G}^+(G + H) = \mathcal{G}^+(G) \oplus \mathcal{G}^+(H) \text{ by Lemma 2.4.1}
$$

$$\begin{aligned}
&= n \oplus m, \\
\mathcal{G}^-(G+H) &= \operatorname{mex}\{\{\mathcal{G}^-(G'+H) \mid G' \text{ an option of } G\}, \\
&\qquad \{\mathcal{G}^-(G+H') \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{\{\mathcal{G}^-(G') \oplus m \mid G' \text{ an option of } G\}, \\
&\qquad \{n \oplus \mathcal{G}^-(H') \mid H' \text{ an option of } H\}\} \\
&\qquad \text{by induction} \\
&= \operatorname{mex}\{0 \oplus m, 1 \oplus m, \cdots, (n-1) \oplus m, \\
&\qquad \{(n+k) \oplus m \mid k \in X \subset \mathbb{N}\}, n \oplus 0, n \oplus 1, \cdots, \\
&\qquad n \oplus (m-1), \{n \oplus (m+j) \mid j \in Y \subset \mathbb{N}\}\},
\end{aligned}$$

for X, Y subsets of $\mathbb{N}$ such that there exists option of G and H, say G' and H' respectively, such that $\mathcal{G}^-(G') = n+k$ and $\mathcal{G}^-(H') = m+j$.

Suppose $(n+k) \oplus m = n \oplus m$. Then, $n+k = n$, so $k = 0$, which implies that there is an option of G, G', with $\mathcal{G}^-(G') = n$, which contradicts $\mathcal{G}^-(G) = n$. Similarly, there is no $j \in \mathbb{N}$ such that $n \oplus (m+j) = n \oplus m$.

Therefore, $\mathcal{G}^-(G+H) = n \oplus m$.

$$\begin{aligned}
\mathcal{G}^-(G+H+\mathbb{2}) &= \operatorname{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H+\mathbb{1}) \\
&\qquad \{\mathcal{G}^-(G'+H+\mathbb{2}) \mid G' \text{ an option of } G\}, \\
&\qquad \{\mathcal{G}^-(G+H'+\mathbb{2}) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{\mathcal{G}^-(G+H), \mathcal{G}^-(G+H) \oplus 1, \\
&\qquad \{\mathcal{G}^-((G'+\mathbb{2})+H) \mid G' \text{ an option of } G\}, \\
&\qquad \{\mathcal{G}^-((H'+\mathbb{2})+G) \mid H' \text{ an option of } H\}\} \\
&\qquad \text{by Lemma 2.1.1} \\
&= \operatorname{mex}\{n \oplus m, n \oplus m \oplus 1, \\
&\qquad \{\mathcal{G}^-((G'+\mathbb{2})+H) \mid G' \text{ an option of } G\}, \\
&\qquad \{\mathcal{G}^-((H'+\mathbb{2})+G) \mid H' \text{ an option of } H\}\} \\
&= \operatorname{mex}\{n \oplus m, n \oplus m \oplus 1, \\
&\qquad \{\mathcal{G}^-((G'+\mathbb{2})) \oplus m \mid G' \text{ an option of } G\},
\end{aligned}$$

$$\begin{aligned}
&\{\mathcal{G}^-(H' + \mathbb{2}) \oplus n \mid H' \text{ an option of } H\}\} \\
&\text{by induction} \\
= \ &\text{mex}\{0 \oplus m \oplus 2, 1 \oplus m \oplus 2, \cdots, (n-1) \oplus m \oplus 2, \\
&\{(n+k) \oplus m \oplus 2 \mid k \in X \subset \mathbb{N}\}, n \oplus 2 \oplus 0, \\
&n \oplus 2 \oplus 1, \cdots, n \oplus 2 \oplus (m-1), \\
&\{n \oplus (m+j) \mid j \in Y \subset \mathbb{N}\}\}.
\end{aligned}$$

Suppose $(n+k)\oplus m\oplus 2 = n\oplus m\oplus 2$. Then $n+k = n$, so $k = 0$, which implies that there is an option of G, G', with $\mathcal{G}^-(G') = n$, contradicting $\mathcal{G}^-(G) = n$. Similarly, there is no $j \in \mathbb{N}$ such that $n \oplus 2 \oplus (m+j) = n \oplus m \oplus 2$.

Therefore $\mathcal{G}^-(G + H + \mathbb{2}) = n \oplus m \oplus 2$.

Thus $\Gamma(G + H) = (n \oplus m)^{(n\oplus m)(n\oplus m\oplus 2)}$.

Therefore, the sum of tame games is tame. ■

Remark. By combining Proposition 2.3.1 and Theorem 2.4.2, given any set of tame games, we can determine the outcome class of their disjunctive sum.

2.4.2 Sum of Tame and Wild Games

Theorem 2.4.2 allows us to easily determine the genus of a sum of tame games without explicitly determining the genera of the options of every position. As of yet, there is no comparable theorem for determining the genus of the sum of two wild games, or even the genus of a sum of a tame and a wild game, except in certain cases, as demonstrated below.

Proposition 2.4.3. *([3], p.431) Suppose G is a game such that $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$. Then $\Gamma(G + \mathbb{1}) = (g \oplus 1)^{(g_0\oplus 1)(g_1\oplus 1)(g_2\oplus 1)(g_3\oplus 1)\cdots}$.*

Proof. Consider $\mathcal{G}^+(G + \mathbb{1})$:

$$\begin{aligned}
\mathcal{G}^+(G + \mathbb{1}) &= \mathcal{G}^+(G) \oplus \mathcal{G}^+(\mathbb{1}) \text{ by Lemma 2.4.1} \\
&= g \oplus 1.
\end{aligned}$$

Take $n \in \mathbb{Z}^{\geq 0}$. Consider

$$\begin{aligned}
\mathcal{G}^-\left(G + \mathbb{1} + \sum_{i=1}^{n} \mathbb{2}\right) &= \mathcal{G}^-\left(\left[G + \sum_{i=1}^{n} \mathbb{2}\right] + \mathbb{1}\right) \\
&= \mathcal{G}^-\left(G + \sum_{i=1}^{n} \mathbb{2}\right) \oplus 1 \text{ by Lemma 2.1.1} \\
&= g_n \oplus 1.
\end{aligned}$$

as required. ■

Proposition 2.4.4. *([3], p.431) Suppose G is a game such that $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$. Then $\Gamma(G + \mathbb{2}) = (g \oplus 2)^{g_1 g_2 g_3 \cdots}$.*

Proof. Consider $\mathcal{G}^+(G + \mathbb{2})$:

$$\begin{aligned}
\mathcal{G}^+(G + \mathbb{2}) &= \mathcal{G}^+(G) \oplus \mathcal{G}^+(\mathbb{2}) \text{ by Lemma 2.4.1} \\
&= g \oplus 2.
\end{aligned}$$

The rest follows since

$$\mathcal{G}^-\left(G + \mathbb{2} + \sum_{i=1}^{n} \mathbb{2}\right) = \mathcal{G}^-\left(G + \sum_{i=1}^{n+1} \mathbb{2}\right),$$

which is precisely what we are trying to show. That is, the n^{th} superscript in the genus of $G + \mathbb{2}$ is the $n + 1^{th}$ superscript in the genus of G. ■

Thus, for any game H which behaves like $\mathbb{1}$ or $\mathbb{2}$, i.e. has only one option to a game with no moves, or two options either to a game whose only option is to a game with no options or to a game with no options, we can easily calculate the genus of the disjunctive sum of H and any other impartial game.

2.4.3 Monoids and Groups of Tame Games

We define the following relation on all impartial misère games :

Definition. Given two games G and H, we say that $G \overset{M}{\equiv} H$ if $\Gamma(G) = \Gamma(H)$.

Proposition 2.4.5. *The relation $\overset{M}{\equiv}$ is an equivalence relation.*

Proof. Let G, H, and K be games.

Reflexivity: Clearly, $\Gamma(G) = \Gamma(G)$, so $G \overset{M}{\equiv} G$.

Symmetry: Suppose, $G \overset{M}{\equiv} H$. Then $\Gamma(G) = \Gamma(H)$. By the symmetry of $=$, $\Gamma(H) = \Gamma(G)$, so $H \overset{M}{\equiv} G$.

Transitivity: Suppose $G \overset{M}{\equiv} H$ and $H \overset{M}{\equiv} K$. Then $\Gamma(G) = \Gamma(H)$, $\Gamma(H) = \Gamma(K)$, and by the transitivity of $=$, $\Gamma(G) = \Gamma(K)$, so $G \overset{M}{\equiv} K$.

■

Notation. Let $\mathcal{T}$ denote the set of tame games. Let $[\mathcal{T}] = \mathcal{T} / \overset{M}{\equiv}$.

We denote the elements of $[\mathcal{T}]$ by their genera. That is, the elements of $[\mathcal{T}]$ are $[0^{120}], [1^{031}]$, etc.

Using Theorem 2.4.2, we obtain the following table of the sums of elements of $[\mathcal{T}]$ ([4], p.137):

$+$	$[0^{120}]$	$[1^{031}]$	$[0^{02}]$	$[1^{13}]$	$[2^{20}]$	$[3^{31}]$	$[4^{46}]$	$[5^{57}]$	$\cdots$
$[0^{120}]$	$[0^{120}]$	$[1^{031}]$	$[0^{02}]$	$[1^{13}]$	$[2^{20}]$	$[3^{31}]$	$[4^{46}]$	$[5^{57}]$	
$[1^{031}]$	$[1^{031}]$	$[0^{120}]$	$[1^{13}]$	$[0^{02}]$	$[3^{31}]$	$[2^{21}]$	$[5^{57}]$	$[4^{46}]$	
$[0^{02}]$	$[0^{02}]$	$[1^{13}]$	$[0^{02}]$	$[1^{13}]$	$[2^{20}]$	$[3^{31}]$	$[4^{46}]$	$[5^{57}]$	
$[1^{13}]$	$[1^{13}]$	$[0^{02}]$	$[1^{13}]$	$[0^{02}]$	$[3^{31}]$	$[2^{20}]$	$[5^{57}]$	$[4^{46}]$	
$[2^{20}]$	$[2^{20}]$	$[3^{31}]$	$[2^{20}]$	$[3^{31}]$	$[0^{02}]$	$[1^{13}]$	$[6^{64}]$	$[7^{75}]$	
$[3^{31}]$	$[3^{31}]$	$[2^{20}]$	$[3^{31}]$	$[2^{20}]$	$[1^{13}]$	$[0^{02}]$	$[7^{75}]$	$[6^{64}]$	
$[4^{46}]$	$[4^{46}]$	$[5^{57}]$	$[4^{46}]$	$[5^{57}]$	$[6^{64}]$	$[7^{75}]$	$[0^{02}]$	$[1^{13}]$	
$[5^{57}]$	$[5^{57}]$	$[4^{46}]$	$[5^{57}]$	$[4^{46}]$	$[7^{75}]$	$[6^{64}]$	$[1^{13}]$	$[0^{02}]$	
$\cdots$									

The chart helps us to illustrate the following:

Theorem 2.4.6. *$[\mathcal{T}]$ forms an Abelian monoid under $+$ with identity $[0^{120}]$.*

Theorem 2.4.7. *The subset of $[\mathcal{T}]$ consisting only of elements of the form $[n^{n(n\oplus 2)}]$ for $n \in \{0, 1, 2, \cdots\}$ forms an Abelian group under $+$ with identity $[0^{02}]$ and each element self-inverse.*

Proof. of Theorems 2.4.6 and 2.4.7 Follows from Theorem 2.4.2. ■

2.5 Future Work

We would like to be able to find some sort of algebraic structure on wild games, in the same way that Theorem 2.4.6 shows there to be an algebraic structure on tame games. Chapter 5 shows that by restricting ourselves to the rules of a fixed game, a periodic structure can sometimes be achieved, but the question remains as to whether there is one in general.

Chapter 3

Subtraction and Taking But Not Breaking Games

In this chapter we will use the genus to analyse impartial heap games where the valid moves are to remove tokens from a heap.

3.1 Subtraction Games

The use of genus gives us the following theorem:

Theorem 3.1.1. *([3], p.442) Every Subtraction game S played under the misère play convention is misère Nim. That is, for h_n a heap of size n played under the rules of S, h_n is tame and $\Gamma(h_n) = 0^{120}$, 1^{031}, or $n^{n(n\oplus 2)}$ for $n \in \{2, 3, 4, \cdots\}$, and each of the options of h_n are tame with genus equal to 0^{120}, 1^{031}, or $n^{n(n\oplus 2)}$ for $n \in \{2, 3, 4, \cdots\}$.*

Proof. Let h_n denote a heap of size n.

We proceed by induction on n.

Consider h_0. Since there are no options, we can easily calculate that the genus of h_0 is 0^{120}. This shows the base case.

Suppose true $\forall$ h_n with $n < k$. Consider $n = k$.

By induction every option of h_k has genus either 0^{120}, 1^{031} or $m^{m(m\oplus 2)}$ for $m \in \{2, 3, 4 \cdots\}$. Denote the genus of h_k by $g^{g_0 g_1 g_2 g_3 \cdots}$.

Suppose $g = 0$. That is $\mathcal{G}^+(h_k) = 0$. By Theorem 3 of ([5]), there exists an option of h_k, say h_m with $m < k$, such that $\mathcal{G}^+(h_m) = 1$. By induction, $\Gamma(h_m) = 1^{031}$.

Examine g_0:

$$g_0 = \text{mex}\{\mathcal{G}^-(h'_k) \mid h'_k \text{ is an option of } h_k\}.$$

If there exists an option of h_k, h_t with $t < n$, such that $\mathcal{G}^-(h_t) = 1$, by induction, $\Gamma(h_t) = 0^{120}$, which contradicts $g = 0$. Since $\mathcal{G}^-(h_m) = 0$, we have that $g_0 = 1$.

Examine g_1:

$$g_1 \quad = \quad \mathcal{G}^-(h_k + 2)$$

$$\begin{aligned} &= \operatorname{mex}\{\mathcal{G}^-(h_k), \mathcal{G}^-(h_k) \oplus 1, \{\mathcal{G}^-(h'_k + 2) \mid h'_k \text{ is an option of } h_k\}\} \\ &\qquad \text{by Proposition 2.1.2} \\ &= \operatorname{mex}\{1, 0, \{\mathcal{G}^-(h'_k + 2) \mid h'_k \text{ is an option of } h_k\}\}. \end{aligned}$$

If there exists an option of h_k, h_t with $t < k$ such that $\mathcal{G}^-(h_t+2) = 2$, by induction, $\Gamma(h_t) = 0^{120}$, which contradicts $g = 0$. Therefore $g_1 = 2$.

Examine g_2:

$$\begin{aligned} g_2 &= \mathcal{G}^-(h_k + 2 + 2) \\ &= \operatorname{mex}\{\mathcal{G}^-(h_k + 2), \mathcal{G}^-(h_k + 2) \oplus 1, \{\mathcal{G}^-(h'_k + 2 + 2) \mid h'_k \text{ is an option of } h_k\}\} \\ &\qquad \text{by Proposition 2.1.2} \\ &= \operatorname{mex}\{2, 3, \{\mathcal{G}^-(h'_k + 2 + 2) \mid h'_k \text{ is an option of } h_k\}\}. \end{aligned}$$

If there exists an option of h_k, h_t with $t < k$ such that $\mathcal{G}^-(h_t + 2 + 2) = 0$, by induction, $\Gamma(h_t) = 0^{120}$, which contradicts $g = 0$. Therefore $g_2 = 0$.

Thus, $\Gamma(h_k) = 0^{120}$.

The cases where $g = 1$ and $g = i \in \{2, 3, 4, \cdots\}$ follow similarly giving $\Gamma(h_k) = 1^{031}$ or $i^{i(i\oplus 2)}$ respectively. Therefore every Subtraction game is misère Nim. ■

Corollary 3.1.2. *For any subtraction game, the genus of h_n depends only on $\mathcal{G}^+(h_n)$.*

Moreover, we know that the $\mathcal{G}^+$ sequence of a finite subtraction game becomes periodic ([1], p.121). That is, there exists $N, p \in \mathbb{N}$ such that $\forall\ n \geq N$, $\mathcal{G}^+(h_n) = \mathcal{G}^+(h_{n+p})$. We then obtain the following theorem:

Theorem 3.1.3. *For any finite subtraction game, the genus sequence of the heaps becomes periodic. That is, for any finite subtraction game, there exists $N, p \in \mathbb{N}$, such that $\forall\ n \geq N$, $\Gamma(h_n) = \Gamma(h_{n+p})$.*

3.2 Taking But Not Breaking Games

In Section 1.2 we defined **Taking and Breaking** games, in which a legal move involved taking tokens from a heap, splitting the heap into smaller heaps, or some combination of the two. We define a **Taking But Not Breaking** game to be one where the only legal move, if there is one available, is to take tokens from a heap. Subtraction games are examples of Taking But Not Breaking Games.

3.3 Subtraction Octals

Definition. A **subtraction octal** is an octal game $0.d_1d_2d_3\cdots$ with $d_i \in \{0, 1, 2, 3\}$ $\forall\ i \in \mathbb{N}$. They are called such as the octal code prohibits splitting of the heaps. A **finite subtraction octal** is a subtraction octal such that there exists $N \in \mathbb{N}$ such that $\forall\ n \geq N$, $d_n = 0$.

Subtraction octals are another example of Taking But Not Breaking games.

In some sense, finite subtraction octals are just finite subtraction games with an initial "pre-sequence". Consider a finite subtraction octal $0.d_1d_2\cdots d_m$. There exists an $i \in \{0, 1, \cdots m\}$ such that $\forall\ j \geq i$, $d_j = 0$ or 3. Then for all heaps of size i and greater, we are playing the subtraction game with subtraction set $\{j \mid d_j = 2 \text{ or } 3\}$.

Example 3.3.1. Consider the finite subtraction octal game 0.23313.

- There is no valid move from a heap with one token.
- We can remove one or two tokens from a heap with two tokens.
- We can remove one, two, or three tokens from a heap with three tokens.
- We can remove one, two, three, or four tokens from a heap with four tokens.
- From a heap with five or more tokens, we can remove one, two, three, or five tokens. That is, for all heaps of size five and greater, we are playing a subtraction game with subtraction set $\{1, 2, 3, 5\}$.

However, even though finite subtraction octal games are a variant on subtraction games, there is no directly comparable theorem to Theorem 3.1.1. In fact, there exist finite subtraction octal games which are not even tame.

Example 3.3.2. Recall the game 0.3122 from Example 2.1.2. The genus of h_4, h_5 and h_7 are not tame. In fact, it can be shown that $\forall\ n \in \mathbb{Z}^{\geq 0}$, the genera of h_{5+7n}, h_{7+7n}, h_{8+7n}, and h_{11+7n} are wild.

There are no wild finite subtraction octal games with octal length two or less. There is one wild finite subtraction octal game with octal length three, 21 with octal length four, 154 with octal length five, and 739 with octal length six. Appendix A lists all wild finite subtraction octal games of octal length six or less.

3.3.1 Periodicity

Since we remove tokens rather than split heaps of tokens, we hope to obtain a comparable theorem to Theorem 3.1.3 for finite subtraction octal games; that is, we would like for any finite subtraction octal game with a heap of size n denoted by h_n, $n \in \mathbb{Z}^{\geq 0}$, there exists $N, p \in \mathbb{N}$, such that $\forall\ n \geq N$, $\Gamma(h_n) = \Gamma(h_{n+p})$. While there are some finite subtraction octals which satisfy this condition, this is not true in general, although there are some notions of periodicity which are true for all finite subtraction octals.

We begin with:

Proposition 3.3.1. *Given a finite subtraction octal game, there exists N, $p \in \mathbb{N}$ such that $\forall\ n \geq N$, $\mathcal{G}^+(h_n) = \mathcal{G}^+(h_{n+p})$. Similarly, for each $v \in \mathbb{Z}^{\geq 0}$, there exists N_v, $p_v \in \mathbb{N}$ such that $\forall\ m \geq N_v$, $\mathcal{G}^-\left(h_m + \sum_{i=1}^{v} 2\right) = \mathcal{G}^-\left(h_{m+p_v} + \sum_{i=1}^{v} 2\right)$.*

Proof. The proof mimics that of Theorem 7.32 of [1].

Consider the finite subtraction octal game $0.d_1 d_2 \cdots d_k$ and $n \geq k+1$. From h_n, there are at most k legal moves. In particular, $\forall\ n \geq k+1$,

$$\mathcal{G}^+(h_n) = \text{mex}\{\mathcal{G}^+(h_{n-i}) \mid d_i = 2 \text{ or } 3\}.$$

Thus $\mathcal{G}^+(h_n) \leq k$, since $|\{\mathcal{G}^+(h_{n-i}) \mid d_i = 2 \text{ or } 3\}| \leq k+1$. That is, $\mathcal{G}^+(h_n) = u$ for $u \in \{0, 1, \cdots, k\}$.

Let $m = \max\{i \mid d_i = 2 \text{ or } 3\}$. That is, m is the largest number of tokens which can be taken from a heap of size n.

The future $\mathcal{G}^+$ sequence from h_n onwards depends only on the previous m values, $\mathcal{G}^+(h_{n-m}), \mathcal{G}^+(h_{n-m+1}), \cdots, \mathcal{G}^+(h_{n-1})$. Not all of these values will be in the mex set which determines $\mathcal{G}^+(h_n)$; the number m is an overestimation assuming that $\forall\ i < m$, $d_i = 3$.

Look at subsequences of length m of the $\mathcal{G}^+$ values. Eventually there will be a subsequence which repeats itself since there are only a finite number of permutations of length m with $k+1$ elements. That is, there exists $p, l \in \mathbb{Z}^{\geq 0}$ such that

$$\mathcal{G}^+(h_n) = \mathcal{G}^+(h_{n+p}) \ \ \forall\ n \text{ such that } l \leq n \leq l+m.$$

Claim $\mathcal{G}^+(h_{n+p}) = \mathcal{G}^+(h_n)\ \forall\ n \geq l$. Proceed by induction on n. We have the base case from the preceding paragraph.

Fix $t \in \mathbb{Z}^{\geq 0}$ and suppose that $\forall\ u < t$,

$$\mathcal{G}^+(h_{(n+u)+p}) = \mathcal{G}^+(h_{n+u}).$$

Consider

$$\begin{aligned}
\mathcal{G}^+(h_{n+t+p}) &= \operatorname{mex}\{\mathcal{G}^+(h_{(n+t+p)-i}) \mid d_i = 2 \text{ or } 3\} \\
&= \operatorname{mex}\{\mathcal{G}^+(h_{(n+t-i)+p}) \mid d_i = 2 \text{ or } 3\} \\
&= \operatorname{mex}\{\mathcal{G}^+(h_{n+t-i}) \mid d_i = 2 \text{ or } 3\} \text{ by induction} \\
&= \operatorname{mex}\{\mathcal{G}^+(h_{(n+t)-i}) \mid d_i = 2 \text{ or } 3\} \\
&= \mathcal{G}^+(h_{n+t}).
\end{aligned}$$

which completes the induction.

Similarly, for each $v \in \mathbb{Z}^{\geq 0}$, the $\mathcal{G}^-(h_m + \sum_{i=1}^{v} 2)$ sequences become periodic, noting that for $v \geq 1$,

$$\begin{aligned}
\mathcal{G}^-\left(h_m + \sum_{i=1}^{v} 2\right) &= \operatorname{mex}\left\{\mathcal{G}^-\left(h_m + \sum_{i=1}^{v-1} 2\right), \mathcal{G}^-\left(h_m + \sum_{i=1}^{v-1} 2\right) \oplus 1,\right. \\
&\qquad \left.\left\{\mathcal{G}^+\left(h_{n-i} + \sum_{i=1}^{v} 2\right) \,\middle|\, d_i = 2 \text{ or } 3\right\}\right\} \\
&\qquad \text{by Proposition 2.1.2.}
\end{aligned}$$

$|\mathcal{G}^-\left(h_m + \sum_{i=1}^{v} 2\right)| \leq k+3$ so $\mathcal{G}^-\left(h_m + \sum_{i=1}^{v} 2\right) \in \{0, 1, \cdots, k+2\}$, so there are at most $k+3$ different possible values for $\mathcal{G}^-\left(h_m + \sum_{i=1}^{v} 2\right)$, rather than $k+1$. However, the rest of the proof follows similarly. ■

Writing the genera of the heaps as follows:

$$\begin{array}{rcccccccc}
\Gamma(h_0) & = & 0 & 1 & 2 & 0 & 2 & 0 & \cdots \\
\Gamma(h_1) & = & a & a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\
\Gamma(h_2) & = & b & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\
\Gamma(h_3) & = & c & c_0 & c_1 & c_2 & c_3 & c_4 & \cdots \\
\Gamma(h_4) & = & d & d_0 & d_1 & d_2 & d_3 & d_4 & \cdots \\
 & \cdots & & & & & & &
\end{array}, \tag{3.1}$$

Proposition 3.3.1 says that each column on the RHS becomes periodic. If there were a column which did not become periodic, then it would be impossible for the genera to become periodic as well.

However, just knowing that each column becomes periodic does not mean that the genera are periodic. If the n^{th} column becomes periodic at the v_n^{th} heap, for $v_1 < v_2 < v_3 < \cdots$, then the genera of the heaps does not become periodic. For example, if the each column becomes periodic where it is indicated in bold:

$$\begin{array}{rcccccccc} \Gamma(h_0) & = & 0 & 1 & \mathbf{2} & 0 & 2 & 0 & \cdots \\ \Gamma(h_1) & = & a & a_0 & \mathbf{a_1} & a_2 & a_3 & a_4 & \cdots \\ \Gamma(h_2) & = & b & \mathbf{b_0} & b_1 & b_2 & b_3 & b_4 & \cdots \\ \Gamma(h_3) & = & c & c_0 & \mathbf{c_1} & c_2 & c_3 & c_4 & \cdots \\ \Gamma(h_4) & = & d & d_0 & d_1 & \mathbf{d_2} & d_3 & d_4 & \cdots \\ & \cdots & & & & & & & \end{array},$$

with the bold pattern continuing diagonally down and to the right, then the genera will never become periodic since the genus of each heaps stabilises at a later digit in the superscript than the index at which the genera stabilises for all previous heaps.

One condition, which, if true, is strong enough to ensure that the genera of the heaps becomes periodic is if there exists $M \in \mathbb{N}$ such that $\forall\ h_n,\ m \geq M$,

$$\mathcal{G}^-\left(h_n + \sum_{i=1}^{m} 2\right) = \mathcal{G}^-\left(h_n + \sum_{i=1}^{m+2} 2\right).$$

That is, there is an index past which the genus of each heap has stabilised. Again, this may not occur, as, recalling the proof of Theorem 2.1.5, the genus can stabilise one, two, or three indices after its options stabilise, but should this occur, then the genera of the heaps becomes periodic with period the least common multiple of the period of each of the columns with index strictly less than M. However, calculationally, if all one wants to show is that the genera for a specific finite subtraction octal game becomes periodic, it is often easier to simply calculate the genera of the first n heaps for n sufficiently large, look for a pattern, and then prove the pattern continues by induction, rather than showing that there exists an M at which each genus stabilises and then determining the period of the first M columns.

There are other abnomolies. The pre-period of each of the columns need not be the same, or even multiples of each other. Equally, the length of the period of each column need not be the same, nor multiples of each other, as the following example shows:

Example 3.3.3. Consider the following finite octal subtraction game S:

$$
\begin{array}{lllllllll}
0.\ 31023 & 21320 & 32002 & 32103 & 20213 & 20231 & 32000 & 00312 & 10213 \\
21031 & 21212 & 12121 & 33300 & 33003 & 13003 & 12031 & 03310 & 10000 \\
00012 & 30000 & 00123 & 00000 & 00000 & 00000 & 00000 & 31233 & 1.
\end{array}
$$

With the use of a computer program written by the author, we obtain

	Pre-period	**Period**
$\mathcal{G}^{+}(S)$	5 683	435
$\mathcal{G}^{-}(S)$	61 139	7 056
$\mathcal{G}^{-}(S+2)$	97 910	14 112
$\mathcal{G}^{-}(S+2+2)$	110 552	28 224
$\mathcal{G}^{-}(S+2+2+2)$	166 438	56 448 .

There is no relationship between the pre-periods and no relationship between the period of $\mathcal{G}^{+}(S)$ and any other periods listed in the above table.

Example 3.3.3 illustrates an interesting (and as of yet, unverifiable) fact, namely that the period lengths of the $\mathcal{G}^{-}$ columns are related. In this case, the period of $\mathcal{G}^{-}(S+2)$ is twice the period of $\mathcal{G}^{-}(S)$, the period of $\mathcal{G}^{-}(S+2+2)$ is four times the period of $\mathcal{G}^{-}(S)$. and the period of $\mathcal{G}^{-}(S+2+2+2)$ is eight times the period of $\mathcal{G}^{-}(S)$. This leads to the following conjecture:

Conjecture 3.3.2. *Let ρ_n be the period of the n^{th} column, thinking of the genera of the heaps as in (3.1), for $n \in \mathbb{N}$. Then there exists an $i \in \mathbb{N}$ such that for each $j \in \mathbb{N}$, there exists $m_j \in \mathbb{N}$ such that $\rho_j = m_j \cdot \rho_i$. That is, there exists a column i such that the period of every other column is some multiple of the period of column i.*

We now return to the question of periodicity of the genus sequence - does there exists $N, p \in \mathbb{N}$, such that $\forall\ n \geq N$, $\Gamma(h_n) = \Gamma(h_{n+p})$? While there is a wealth of evidence to support the conjecture, such as 0.123 (whose genera have period five with pre-period four), 0.21123 (whose genera have period forty-eight with no pre-period), 0.100213 (whose genera have period ten with pre-period seven), 0.33121 (whose genera have period nine with pre-period three), and 0.01023 (whose genera have period nine with pre-period seven), there is also a counterexample - 0.122 213.

Notation. Let $\{a_1a_2\cdots a_n\}^m$ denote the string $a_1a_2\cdots a_n$ repeated m times.

Example 3.3.4.

$$\{a_1a_2a_3\}^6 = a_1a_2a_3\ a_1a_2a_3\ a_1a_2a_3\ a_1a_2a_3\ a_1a_2a_3\ a_1a_2a_3$$

Proposition 3.3.3. *Let $G = 0.122\ 213$. For $n \in \mathbb{Z}^{\geq 0}$, let h_n denote a heap of size n. Then*

n	*genus*
h_{24+40n}	$2^{\{\{43131\}^2\{42020\}^2\}^n 43131\ 431}$
h_{25+40n}	$2^{131\{43131\{42020\}^2 43131\}^n 431}$
h_{26+40n}	$0^{0\{43131\{42020\}^2 43131\}^n 43131\ 420}$
h_{27+40n}	$0^{3131\{\{42020\}^2\{43131\}^2\}^n 420}$
h_{28+40n}	$3^{31\{\{42020\}^2\{43131\}^2\}^n 420}$
h_{29+40n}	$1^{\{\{42020\}^2\{43131\}^2\}^n 42020\ 420}$
h_{30+40n}	$1^{120\{42020\{43131\}^2 42020\}^n 42020\ 431}$
h_{31+40n}	$4^{0\{42020\{43131\}^2 42020\}^n 42020\ 431}$
h_{32+40n}	$2^{2020\{\{43131\}^2\{42020\}^2\}^n 43131\ 431}$
h_{33+40n}	$2^{20\{\{43131\}^2\{42020\}^2\}^n 43131\ 431}$
h_{34+40n}	$0^{\{\{43131\}^2\{42020\}^2\}^n \{43131\}^2 420}$
h_{35+40n}	$0^{13143\ 131\{\{42020\}^2\{43131\}^2\}^n 420}$
h_{36+40n}	$3^{04313\ 1\{\{42020\}^2\{43131\}^2\}^n 42020\ 420}$
h_{37+40n}	$1^{3131\{\{42020\}^2\{43131\}^2\}^n 42020\ 420}$
h_{38+40n}	$1^{31\{\{42020\}^2\{43131\}^2\}^n \{42020\}^2 431}$
h_{39+40n}	$4^{\{\{42020\}^2\{43131\}^2\}^n \{42020\}^2 431}$
h_{40+40n}	$2^{12042\ 020\{\{43131\}^2\{42020\}^2\}^n 43131\ 431}$
	continued on next page

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n	*genus*
h_{41+40n}	$2^{04202\ 0\left\{\{43131\}^2\{42020\}^2\right\}^n 43131\ 431}$
h_{42+40n}	$0^{2020\left\{\{43131\}^2\{42020\}^2\right\}^n\{43131\}^2 420}$
h_{43+40n}	$0^{20\{\{43131\}\{42020\}\}^n\{43131\}^2 420}$
h_{44+40n}	$3^{\left\{\{43131\}^2\{42020\}^2\right\}^n\{43131\}^2 42020\ 420}$
h_{45+40n}	$1^{13143\ 131\left\{\{42020\}^2\{43131\}^2\right\}^n 42020\ 420}$
h_{46+40n}	$1^{04313\ 1\left\{\{42020\}^2\{43131\}^2\right\}^n\{42020\}^2 431}$
h_{47+40n}	$4^{3131\left\{\{42020\}^2\{43131\}^2\right\}^n\{42020\}^2 431}$
h_{48+40n}	$2^{31\left\{\{42020\}^2\{43131\}^2\right\}^n\{42020\}^2 43131\ 431}$
h_{49+40n}	$2^{\left\{\{42020\}^2\{43131\}^2\right\}^n\{42020\}^2 43131\ 431}$
h_{50+40n}	$0^{12042\ 020\left\{\{43131\}^2\{42020\}^2\right\}^n\{43131\}^2 420}$
h_{51+40n}	$0^{04202\ 0\left\{\{43131\}^2\{42020\}^2\right\}^n\{43131\}^2 420}$
h_{52+40n}	$3^{2020\left\{\{43131\}^2\{42020\}^2\right\}^n\{43131\}^2 42020\ 420}$
h_{53+40n}	$1^{20\left\{\{43131\}^2\{42020\}^2\right\}^n\{43131\}^2 42020\ 420}$
h_{54+40n}	$1^{\left\{\{43131\}^2\{42020\}^2\right\}^{n+1} 431}$
h_{55+40n}	$4^{131\left\{\{43131\}^2\{42020\}^2\right\}^n\{42020\}^2 431}$
h_{56+40n}	$2^{0\left\{43131\{42020\}^2 43131\right\}^{n+1} 431}$
h_{57+40n}	$2^{3131\left\{\{42020\}^2\{43131\}^2\right\}^n\{42020\}^2 43131\ 431}$
h_{58+40n}	$0^{31\left\{\{42020\}^2\{43131\}^2\right\}^{n+1} 420}$
h_{59+40n}	$0^{\left\{\{42020\}^2\{43131\}^2\right\}^{n+1} 420}$
h_{60+40n}	$3^{120\left\{42020\{43131\}^2 42020\right\}^{n+1} 420}$
h_{61+40n}	$1^{0\left\{42020\{43131\}^2 42020\right\}^{n+1} 420}$
h_{62+40n}	$1^{2020\left\{\{43131\}^2\{42020\}^2\right\}^{n+1} 431}$
h_{63+40n}	$4^{20\left\{\{43131\}^2\{42020\}^2\right\}^{n+1} 431}$

Proof. We proceed by induction on n.

For $n = 0$, calculations give us

n	genus
h_{24}	$2^{43131\ 431}$
	continued on next page

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n	genus
h_{25}	$2^{13143\ 1}$
h_{26}	$0^{04313\ 1420}$
h_{27}	$0^{31314\ 20}$
h_{28}	$3^{31420\ 20420}$
h_{29}	$1^{42020\ 420}$
h_{30}	$1^{12042\ 02043\ 1}$
h_{31}	$4^{04202\ 0431}$
h_{32}	$2^{20204\ 31314\ 31}$
h_{33}	$2^{20431\ 31431}$
h_{34}	$0^{43131\ 43131\ 420}$
h_{35}	$0^{13143\ 13142\ 0}$
h_{36}	$3^{04313\ 14202\ 0420}$
h_{37}	$1^{31314\ 20204\ 20}$
h_{38}	$1^{31420\ 20420\ 20431}$
h_{39}	$4^{42020\ 42020\ 431}$
h_{40}	$2^{12042\ 02043\ 13143\ 1}$
h_{41}	$2^{04202\ 04313\ 1431}$
h_{42}	$0^{20204\ 31314\ 31314\ 20}$
h_{43}	$0^{20431\ 31431\ 31420}$
h_{44}	$3^{43131\ 43131\ 42020\ 420}$
h_{45}	$1^{13143\ 13142\ 02042\ 0}$
h_{46}	$1^{04313\ 14202\ 04202\ 0431}$
h_{47}	$4^{31314\ 20204\ 20204\ 31}$
h_{48}	$2^{31420\ 20420\ 20431\ 31431}$
h_{49}	$2^{42020\ 42020\ 43131\ 431}$
h_{50}	$0^{12042\ 02043\ 13143\ 13142\ 0}$
h_{51}	$0^{04202\ 04313\ 14313\ 1420}$
h_{52}	$3^{20204\ 31314\ 31314\ 20204\ 20}$

continued on next page

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n	genus
h_{53}	$1^{20431\ 31431\ 31420\ 20420}$
h_{54}	$1^{43131\ 43131\ 42020\ 42020\ 431}$
h_{55}	$4^{13143\ 13142\ 02042\ 02043\ 1}$
h_{56}	$2^{04313\ 14202\ 04202\ 04313\ 1431}$
h_{57}	$2^{31314\ 20204\ 20204\ 31314\ 31}$
h_{58}	$0^{31420\ 20420\ 20431\ 31431\ 31420}$
h_{59}	$0^{42020\ 42020\ 43131\ 43131\ 420}$
h_{60}	$3^{12042\ 02043\ 13143\ 13142\ 02042\ 0}$
h_{61}	$1^{04202\ 04313\ 14313\ 14202\ 0420}$
h_{62}	$1^{20204\ 31314\ 31314\ 20204\ 20204\ 31}$
h_{63}	$4^{20431\ 31431\ 31420\ 20420\ 20431}$

which shows the base case.

Suppose true $\forall\ n < k$. That is, for all $n < k$, the genus of a heap of size h_{i+40n}, for $i \in \{24, 25, \cdots, 63\}$, equals the genus given in the chart in the statement of the theorem. Call this (IH1).

Consider $n = k$.

We will only show the result for h_{24+40k}, as the method of proof is similar for all heaps.

The moves available from h_{24+40k} are

$$\begin{aligned} h_{24+40k} &\xrightarrow{-2} h_{24+40k-2} = h_{62+40(k-1)} \\ &\xrightarrow{-3} h_{24+40k-3} = h_{61+40(k-1)} \\ &\xrightarrow{-4} h_{24+40k-4} = h_{60+40(k-1)} \\ &\xrightarrow{-6} h_{24+40k-6} = h_{58+40(k-1)}, \end{aligned}$$

where each of the options falls under the induction hypothesis. That is

$$\begin{aligned} \Gamma\left(h_{61+40(k-1)}\right) &= 1^{2020\left\{\{43131\}^2\{42020\}^2\right\}^k 431}, \\ \Gamma\left(h_{61+40(k-1)}\right) &= 1^{0\left\{42020\{43131\}^2 42020\right\}^k 420}, \end{aligned}$$

$$\begin{aligned}
\Gamma\left(h_{60+40(k-1)}\right) &= 3^{120\left\{42020\{43131\}^2 42020\right\}^k 420}, \\
\Gamma\left(h_{58+40(k-1)}\right) &= 0^{31\left\{\{42020\}^2\{43131\}^2\right\}^k 420}.
\end{aligned}$$

We want

$$\Gamma\left(h_{24+40k}\right) = 2^{\left\{\{43131\}^2\{42020\}^2\right\}^k 43131\ 431}. \tag{3.2}$$

$$\begin{aligned}
\mathcal{G}^+(h_{24+40k}) &= \operatorname{mex}\{1,1,3,0\} \\
&= 2, \\
\mathcal{G}^-(h_{24+40k}) &= \operatorname{mex}\{2,0,1,3\} \\
&= 4.
\end{aligned}$$

Therefore the base and the first superscript equal the desired result.

Consider

$$\mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{m} 2\right)$$

for $m \in \mathbb{N}$. Claim that this equals the $(m+1)^{th}$ exponent of the genus on the RHS of Equation (3.2)

Proceed by induction on m.

Suppose $m = 1$. Then

$$\begin{aligned}
\mathcal{G}^-(h_{24+40k} + 2) &= \operatorname{mex}\{\mathcal{G}^-(h_{24+40k}), \mathcal{G}^-(h_{24+40k}) \oplus 1, \mathcal{G}^-(h_{62+40(k-1)} + 2), \\
&\quad \mathcal{G}^-(h_{61+40(k-1)} + 2), \mathcal{G}^-(h_{60+40(k-1)} + 2), \mathcal{G}^-(h_{58+40(k-1)} + 2)\} \\
&\quad \text{by Proposition 2.1.2} \\
&= \operatorname{mex}\{4, 5, \mathcal{G}^-(h_{24+40k}) \oplus 1, \mathcal{G}^-(h_{62+40(k-1)} + 2), \\
&\quad \mathcal{G}^-(h_{61+40(k-1)} + 2), \mathcal{G}^-(h_{60+40(k-1)} + 2), \mathcal{G}^-(h_{58+40(k-1)} + 2)\} \\
&= \operatorname{mex}\{4,5,0,4,2,1\} \text{ by (IH1)} \\
&= 3,
\end{aligned}$$

as required.

Now suppose true $\forall\ m < 10k+6$. That is

$$\Gamma\left(h_{24+40k}\right) = 2^{\left\{\{43131\}^2\{42020\}^2\right\}^n 43131\ 4\ g_{10k+6}\ g_{10k+7}\ g_{10k+8}\cdots} \tag{3.3}$$

for $g_{10k+i} \in \mathbb{Z}^{\geq 0}$, $i \in \{6, 7, \cdots\}$.

Examine g_{10k+6}:

$$\begin{aligned}
g_{10k+6} &= \mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+6} 2\right) \\
&= \operatorname{mex}\left\{\mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+5} 2\right), \mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+5} 2\right) \oplus 1,\right. \\
&\quad \mathcal{G}^-\left(h_{62+40(k-1)} + \sum_{i=1}^{10k+6} 2\right), \mathcal{G}^-\left(h_{61+40(k-1)} + \sum_{i=1}^{10k+6} 2\right), \\
&\quad \left.\mathcal{G}^-\left(h_{60+40(k-1)} + \sum_{i=1}^{10k+6} 2\right), \mathcal{G}^-\left(h_{58+40(k-1)} + \sum_{i=1}^{10k+6} 2\right)\right\} \\
&\quad \text{by Proposition 2.1.2} \\
&= \operatorname{mex}\left\{4, 5, \mathcal{G}^-\left(h_{62+40(k-1)} + \sum_{i=1}^{10k+6} 2\right), \mathcal{G}^-\left(h_{61+40(k-1)} + \sum_{i=1}^{10k+6} 2\right),\right. \\
&\quad \left.\mathcal{G}^-\left(h_{60+40(k-1)} + \sum_{i=1}^{10k+6} 2\right), \mathcal{G}^-\left(h_{58+40(k-1)} + \sum_{i=1}^{10k+6} 2\right)\right\} \\
&\quad \text{by Equation (3.3)} \\
&= \operatorname{mex}\{4, 5, 1, 2, 2, 0\} \text{ by (IH1)} \\
&= 3,
\end{aligned}$$

as required.

Examine g_{10k+7}:

$$\begin{aligned}
g_{10k+7} &= \mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+7} 2\right) \\
&= \operatorname{mex}\left\{\mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+6} 2\right), \mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+6} 2\right) \oplus 1,\right. \\
&\quad \mathcal{G}^-\left(h_{62+40(k-1)} + \sum_{i=1}^{10k+7} 2\right), \mathcal{G}^-\left(h_{61+40(k-1)} + \sum_{i=1}^{10k+7} 2\right), \\
&\quad \left.\mathcal{G}^-\left(h_{60+40(k-1)} + \sum_{i=1}^{10k+7} 2\right), \mathcal{G}^-\left(h_{58+40(k-1)} + \sum_{i=1}^{10k+7} 2\right)\right\} \\
&\quad \text{by Proposition 2.1.2} \\
&= \operatorname{mex}\left\{3, 2, \mathcal{G}^-\left(h_{62+40(k-1)} + \sum_{i=1}^{10k+7} 2\right), \mathcal{G}^-\left(h_{61+40(k-1)} + \sum_{i=1}^{10k+7} 2\right),\right.
\end{aligned}$$

$$
\begin{aligned}
& \quad \mathcal{G}^-\left(h_{60+40(k-1)} + \sum_{i=1}^{10k+7} 2\right), \mathcal{G}^-\left(h_{58+40(k-1)} + \sum_{i=1}^{10k+7} 2\right)\Bigg\} \\
& \quad \text{by Equation (3.3)} \\
&= \operatorname{mex}\{3, 2, 3, 0, 0, 2\} \text{ by (IH1)} \\
&= 1,
\end{aligned}
$$

as required.

Examine g_{10k+8}:

$$
\begin{aligned}
g_{10k+8} &= \mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+8} 2\right) \\
&= \operatorname{mex}\Bigg\{\mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+7} 2\right), \mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+7} 2\right) \oplus 1, \\
& \quad \mathcal{G}^-\left(h_{62+40(k-1)} + \sum_{i=1}^{10k+8} 2\right), \mathcal{G}^-\left(h_{61+40(k-1)} + \sum_{i=1}^{10k+8} 2\right), \\
& \quad \mathcal{G}^-\left(h_{60+40(k-1)} + \sum_{i=1}^{10k+8} 2\right), \mathcal{G}^-\left(h_{58+40(k-1)} + \sum_{i=1}^{10k+8} 2\right)\Bigg\} \\
& \quad \text{by Proposition 2.1.2} \\
&= \operatorname{mex}\Bigg\{1, 0, \mathcal{G}^-\left(h_{62+40(k-1)} + \sum_{i=1}^{10k+8} 2\right), \mathcal{G}^-\left(h_{61+40(k-1)} + \sum_{i=1}^{10k+8} 2\right), \\
& \quad \mathcal{G}^-\left(h_{60+40(k-1)} + \sum_{i=1}^{10k+8} 2\right), \mathcal{G}^-\left(h_{58+40(k-1)} + \sum_{i=1}^{10k+8} 2\right)\Bigg\} \\
& \quad \text{by Equation (3.3)} \\
&= \operatorname{mex}\{1, 0, 1, 2, 2, 0\} \text{ by (IH1)} \\
&= 3,
\end{aligned}
$$

as required.

By (IH1), the genus of each of the options has stabilised by this index. Since the genus of h_{24+40k} has exhibited stabilising behaviour, this gives

$$h_{24+40k} = 2^{\left\{\{43131\}^2\{42020\}^2\right\}^n 43131\ 431}$$

as required.

The other thirty-nine cases follow similarly. $\blacksquare$

Theorem 3.3.4. *Let* $G = 0.122\ 213$. *Then the genus sequence of the heaps of* G *never becomes periodic.*

Proof. Consider heaps h_{24+40n} for $n \in \mathbb{Z}^{\geq 0}$. By Proposition 3.3.3,

$$\Gamma\left(h_{24+40n}\right) = 2^{\left\{\{43131\}^2\{42020\}^2\right\}^n 43131\ 431},$$

and for heaps of size twenty-four or greater, the only heaps whose genera have 2^{43} as its starting digits are heaps of the form h_{24+40k} for some $k \in \mathbb{Z}^{\geq 0}$. Thus, if the genus sequence of the heaps of this game were to be periodic, there exists $N \in \mathbb{Z}^{\geq 0}$, such that for $n \geq N$, there exists $n_1, n_2, \cdots n_i, \cdots \in \mathbb{Z}^{\geq 0}$ such that

$$\Gamma\left(h_{24+40n}\right) = \Gamma\left(h_{24+40n_1}\right) = \Gamma\left(h_{24+40n_2}\right) = \cdots = \Gamma\left(h_{24+40n_i}\right) = \cdots.$$

Claim that for $n \in \mathbb{Z}^{\geq 0}$, there does not exist $m \in \mathbb{Z}^{\geq 0}$, $m \neq n$, such that $\Gamma\left(h_{24+40n}\right) = \Gamma\left(h_{24+40m}\right)$.

Fix n, $m \in \mathbb{Z}^{\geq 0}$, $n \neq m$, and suppose, without loss of generality, that $n < m$. By Proposition 3.3.3,

$$\mathcal{G}^-\left(h_{24+40m} + \sum_{i=1}^{10m+5}\right) = 4,$$

while

$$\mathcal{G}^-\left(h_{24+40n} + \sum_{i=1}^{10m+5}\right) = 1.$$

Therefore the genera of h_{24+40m} and h_{24+40n} cannot be the same if $n \neq m$ as digits in the superscript are not all equal.

Therefore the genus sequence of the heaps of G never become periodic. ■

Again, this, unfortunately, shows how finite subtraction octals do not necessarily behave like pure finite subtraction games, which, by Theorem 3.1.3, have a periodic genus sequence.

3.3.2 The Domestication of Finite Subtraction Octal Games

Definition. Given a wild finite subtraction octal game $G = 0.d_1 d_2 \cdots d_n$, we **nimify** G by appending an infinite number of threes to the end of the octal, i.e. $0.d_1 d_2 \cdots d_n \bar{3}$. We call the resulting octal $\mathcal{N}(G)$.

Recall from Example 1.2.2 that the octal code for Nim is $0.\bar{3}$; by nimifying G, we are trying to make G's octal code look much like Nim's as we possibly can in the hopes that $\mathcal{N}(G)$ will now behave like Nim.

Definition. Given a wild finite subtraction octal game G, we say that G is **domesticatable** if $\forall\ n \in \mathbb{Z}^{\geq 0}$, for h_n a heap of $\mathcal{N}(G)$, h_n is tame.

In essence, if G is domesticatable, then nimifying worked. That is, by making the octal code of G resemble Nim's, G now behaves like Nim. Not all wild finite subtraction octal games are domesticatable however - see Appendix A for a list of non-domesticatable finite subtraction octal games of octal length six or less.

Some wild finite subtraction octal games almost take to being nimified but are still a bit too wild.

Definition. Given a wild finite subtraction octal game G, we say that G is **almost domesticatable** if there exists $N \in \mathbb{N}$ such that there exists at least one $m \in \mathbb{N}$, $m < N$, h_m a heap of $\mathcal{N}(G)$ such that h_m is wild and $\forall\ n \geq N$, for h_n a heap of $\mathcal{N}(G)$, $\Gamma\left(h_n\right) = 0^{120}, 1^{031}$, or $n^{n(n\oplus 2)}$ for $n \in \mathbb{Z}^{\geq 0}$. That is, G is almost domesticatable if $\mathcal{N}(G)$ has only a finite number of heaps whose genera equal wild values.

We need to be careful to note that for an almost domesticatable finite subtraction octal game, the large heaps are not tame, rather they have their genus equal to a tame value, as for a game to be tame, its options must also be tame, and for large heaps in the nimified game, amongst a heap's options will be the wild heaps. Frequently, supposing $0.d_1d_2\cdots d_n$ is a almost domesticatable finite subtraction octal, wild heaps occur at heap size m for $m \leq n$ where the nimification of the game has no effect. Appendix A makes note of wild finite subtraction octals of length six or less which are almost domesticatable.

Definition. Given a finite octal game G, the **Nimming Number** is the least number of threes which must be appended to the octal code such that the resulting game is tame.

A tame finite subtraction octal has Nimming Number 0, since we need not append any threes to obtain a tame game. A wild finite subtraction octal game in which appending any number of threes (including an infinite number) does not make the resulting game tame is said to have an undefined Nimming Number. A wild finite subtraction octal game in which the game only becomes tame if an infinite number of threes are appended is said to have Nimming Number ∞. In all finite subtraction octal games examined, the Nimming Number is either undefined or ∞, which leads to the following conjecture:

Conjecture 3.3.5. *Let G be a finite subtraction octal game. Then the Nimming Number of G is either undefined or ∞.*

We conclude with three proofs, one proving that 0.123 is domesticatable, one proving 0.123 has Nimming Number ∞, and one proving that 0.3103 is not domesticatable. These proofs are representative of how one shows that a specific wild finite subtraction octal game is domesticatable or not domesticatable, as well as the calculation of the Nimming Number.

Proposition 3.3.6. 0.123 *is domesticatable.*

Proof. Let h_n denote a heap of size n in the game $\mathcal{N}(0.123)$.

Claim

$$\Gamma(h_n) = \begin{cases} 0^{120} & \text{if } n = 0, 2 \\ 1^{031} & \text{if } n = 1 \\ \lceil \frac{n}{2} \rceil^{\lceil \frac{n}{2} \rceil (\lceil \frac{n}{2} \rceil \oplus 2)} & \text{else.} \end{cases}$$

We proceed by induction on n.

The genera of h_0, h_1, h_2, and h_3 follows from their genera in the game 0.123. They are

heap	genus
h_0	0^{120}
h_1	1^{031}
h_2	0^{120}
h_3	2^{20}

Consider h_4. There are three moves from h_4:

$$\begin{aligned} h_4 &\xrightarrow{-4} h_0 \\ h_4 &\xrightarrow{-3} h_1 \\ h_4 &\xrightarrow{-2} h_2. \end{aligned}$$

By Proposition 2.1.2, $\Gamma(h_4) = 2^{20}$.

Suppose true $\forall\ n < 2k$. Consider $n = 2k$. There are $2k - 1$ moves from h_{2k}:

$$\begin{aligned} h_{2k} &\xrightarrow{-2k} h_0 \\ h_{2k} &\xrightarrow{-2k+1} h_1 \\ h_{2k} &\xrightarrow{-2k+2} h_2 \\ &\cdots \\ h_{2k} &\xrightarrow{4} h_{2k-4} \\ h_{2k} &\xrightarrow{3} h_{2k-3} \\ h_{2k} &\xrightarrow{2} h_{2k-2}. \end{aligned}$$

By induction, their respective genera are:

heap	genus
h_0	0^{120}
h_1	1^{031}
h_2	0^{120}
h_3	2^{20}
$\cdots$	
h_{2k-4}	$(k-2)^{(k-2)((k-2)\oplus 2)}$
h_{2k-3}	$(k-1)^{(k-1)((k-1)\oplus 2)}$
h_{2k-2}	$(k-1)^{(k-1)((k-1)\oplus 2)}$

By Proposition 2.1.2, $\Gamma(h_{2k}) = k^{k(k\oplus 2)}$, which is tame. Since $\left\lceil \frac{2k}{2} \right\rceil = k$, we have shown the result for even numbers.

Similarly, $\Gamma(h_{2k+1}) = (k+1)^{(k+1)((k+1)\oplus 2)}$. Since $\left\lceil \frac{2k+1}{2} \right\rceil = k+1$, this shows the result for odd numbers.

Therefore 0.123 is domesticatable. ∎

Proposition 3.3.7. *The Nimming Number of* 0.123 *is* ∞*. That is* $0.123(3)^n$ *is wild for any* $n \in \mathbb{Z}^{\geq 0}$*.*

Proof. Fix $n \in \mathbb{N}$. Examine the game $0.123(3)^n$. Let h_n denote a heap of size n in the game $0.123(3)^n$. By Proposition 3.3.6, we know that all heaps up to size $n+3$ are tame.

Consider h_{n+4}. The moves available from h_{n+4} are:

$$\begin{aligned}
h_{n+4} &\xrightarrow{-(n+3)} h_1\\
h_{n+4} &\xrightarrow{-(n+2)} h_2\\
h_{n+4} &\xrightarrow{-(n+1)} h_3\\
&\cdots\\
h_{n+4} &\xrightarrow{-4} h_n\\
h_{n+4} &\xrightarrow{-3} h_{n+1}\\
h_{n+4} &\xrightarrow{-2} h_{n+2}.
\end{aligned}$$

By Proposition 3.3.6, their respective genera are:

heap	genus
h_0	0^{120}
h_1	1^{031}
h_2	0^{120}
h_3	2^{20}
$\cdots$	
h_n	$\lceil \frac{n}{2} \rceil^{\lceil \frac{n}{2} \rceil (\lceil \frac{n}{2} \rceil \oplus 2)}$
h_{n+1}	$\lceil \frac{n+1}{2} \rceil^{\lceil \frac{n+1}{2} \rceil (\lceil \frac{n+1}{2} \rceil \oplus 2)}$
h_{n+2}	$\lceil \frac{n+2}{2} \rceil^{\lceil \frac{n+2}{2} \rceil (\lceil \frac{n+2}{2} \rceil \oplus 2)}$

By Proposition 2.1.2, $\Gamma(h_{n+4}) = \lceil \frac{n+4}{2} \rceil^{\lceil \frac{n+4}{2} \rceil (\lceil \frac{n+4}{2} \rceil \oplus 2)}$.

Claim the genus of h_{n+5} is 1^{031}. The moves available from h_{n+5} are

$$\begin{aligned}
h_{n+5} &\xrightarrow{-(n+3)} h_2\\
h_{n+5} &\xrightarrow{-(n+2)} h_3\\
h_{n+5} &\xrightarrow{-(n+1)} h_4
\end{aligned}$$

$$\begin{array}{lcl} \dots & & \\ h_{n+5} & \xrightarrow{-4} & h_{n+1} \\ h_{n+5} & \xrightarrow{-3} & h_{n+2} \\ h_{n+5} & \xrightarrow{-2} & h_{n+3}. \end{array}$$

The only heap with genus equalling 1^{031} with size less than $n+3$ is h_1, which is not an option of h_{n+5}. Moreover, $\Gamma(h_2) = 0^{120}$. By Proposition 2.1.2, $\Gamma(h_{n+5}) = 1^{031}$.

Claim $\Gamma(h_{n+6}) = 0^{02}$. The moves available from h_{n+6} are

$$\begin{array}{lcl} h_{n+6} & \xrightarrow{-(n+3)} & h_3 \\ h_{n+6} & \xrightarrow{-(n+2)} & h_4 \\ h_{n+6} & \xrightarrow{-(n+1)} & h_5 \\ \dots & & \\ h_{n+6} & \xrightarrow{-4} & h_{n+2} \\ h_{n+6} & \xrightarrow{-3} & h_{n+3} \\ h_{n+6} & \xrightarrow{-2} & h_{n+4}. \end{array}$$

By preceding work, their respective genera are:

heap	genus
h_3	2^{20}
h_4	2^{20}
h_5	3^{31}
$\dots$	
h_{n+2}	$\left\lceil \frac{n+2}{2} \right\rceil^{\left\lceil \frac{n+2}{2} \right\rceil \left(\left\lceil \frac{n+2}{2} \right\rceil \oplus 2\right)}$
h_{n+3}	$\left\lceil \frac{n+3}{2} \right\rceil^{\left\lceil \frac{n+3}{2} \right\rceil \left(\left\lceil \frac{n+3}{2} \right\rceil \oplus 2\right)}$
h_{n+4}	$\left\lceil \frac{n+4}{2} \right\rceil^{\left\lceil \frac{n+4}{2} \right\rceil \left(\left\lceil \frac{n+4}{2} \right\rceil \oplus 2\right)}$

By Proposition 2.1.2, $\Gamma(h_{n+6}) = 0^{02}$.

Claim h_{n+8} is wild. The moves available from h_{n+8} are:

$$\begin{array}{lcl} h_{n+8} & \xrightarrow{-(n+3)} & h_5 \\ h_{n+8} & \xrightarrow{-(n+2)} & h_6 \end{array}$$

$$\begin{array}{ccc}
h_{n+8} & \xrightarrow{-(n+1)} & h_7 \\
\cdots & & \\
h_{n+8} & \xrightarrow{-4} & h_{n+4} \\
h_{n+8} & \xrightarrow{-3} & h_{n+5} \\
h_{n+8} & \xrightarrow{-2} & h_{n+6}.
\end{array}$$

By preceding work, their respective genera are:

heap	genus
h_5	3^{31}
h_6	3^{31}
h_7	4^{46}
$\cdots$	
h_{n+4}	$\left\lceil\frac{n+4}{2}\right\rceil^{\left\lceil\frac{n+4}{2}\right\rceil\left(\left\lceil\frac{n+4}{2}\right\rceil\oplus 2\right)}$
h_{n+5}	1^{031}
h_{n+6}	0^{02}

Since all options of the position are tame, and there is an option with genus 0^{02} and an option with 1^{031}, and no options with genera 0^{120} or 1^{13}, by Theorem 2.2.2, h_{n+8} is wild. By Proposition 2.1.2, $\Gamma\left(h_{n+8}\right) = 2^{1420}$. ∎

Proposition 3.3.8. *The finite subtraction octal game 0.3103 is not domesticatable.*

Proof. Let h_n denote a heap of size n in the game $\mathcal{N}(0.3103)$

The genera of heaps of size zero through three are:

heap	genus
h_0	0^{120}
h_1	1^{031}
h_2	2^{20}
h_3	0^{02}

Claim for $n \in \mathbb{N}$,

$$\begin{aligned}
\Gamma\left(h_{4n}\right) &= (2n-1)^{(2n)(2n\oplus 2)}, \\
\Gamma\left(h_{4n+1}\right) &= 2n^{(2n+1)((2n+1)\oplus 2)},
\end{aligned}$$

$$\begin{aligned}\Gamma\left(h_{4n+2}\right) &= (2n+1)^{(2n+2)((2n+2)\oplus 2)},\\ \Gamma\left(h_{4n+3}\right) &= (2n+2)^{(2n+1)((2n+1)\oplus 2)}.\end{aligned}$$

Proceed by induction on n.

Calculations give that the genera of heaps of size four through seven are:

heap	genus
h_4	1^{20}
h_5	2^{31}
h_6	3^{46}
h_7	4^{31}

which shows the base case.

Suppose now true $\forall\ n < k$. Consider $n = k$. We will show that $\Gamma\left(h_{4k}\right) = (2k-1)^{(2k)(2k\oplus 2)}$, since the proofs for $4k+1$, $4k+2$, and $4k+3$ are similar and relatively straightforward.

The moves available from h_{4k} are:

$$\begin{array}{lcl}
h_{4k} & \xrightarrow{-(4k)} & h_0\\
h_{4k} & \xrightarrow{-(4k-1)} & h_1\\
h_{4k} & \xrightarrow{-(4k-2)} & h_2\\
h_{4k} & \xrightarrow{-(4k-3)} & h_3\\
\cdots & & \\
h_{4k} & \xrightarrow{-8} & h_{4k-8}\\
h_{4k} & \xrightarrow{-7} & h_{4k-7}\\
h_{4k} & \xrightarrow{-6} & h_{4k-6}\\
h_{4k} & \xrightarrow{-5} & h_{4k-5}\\
h_{4k} & \xrightarrow{-4} & h_{4k-4}\\
h_{4k} & \xrightarrow{-1} & h_{4k-1}.
\end{array}$$

We now begin the calculations:

$$\mathcal{G}^{+}(h_{4k}) = \operatorname{mex}\{\mathcal{G}^{+}(h_0), \mathcal{G}^{+}(h_1), \mathcal{G}^{+}(h_2), \mathcal{G}^{+}(h_3), \cdots,$$

$$
\begin{aligned}
&\quad \mathcal{G}^+(h_{4k-8}), \mathcal{G}^+(h_{4k-7}), \mathcal{G}^+(h_{4k-6}),\\
&\quad \mathcal{G}^+(h_{4k-5}), \mathcal{G}^+(h_{4k-4}), \mathcal{G}^+(h_{4k-1})\}\\
&= \operatorname{mex}\{0,1,2,0,\cdots,2(k-2)-1,2(k-2),2(k-2)+1,\\
&\quad 2(k-2)+2,2(k-1)-1,2(k-1)+2\} \text{ by induction}\\
&= \operatorname{mex}\{0,1,2,0,\cdots,2k-5,2k-4,2k-3,2k-2,2k-3,2k\}\\
&= 2k-1,\\
\mathcal{G}^-(h_{4k}) &= \operatorname{mex}\{\mathcal{G}^-(h_0), \mathcal{G}^-(h_1), \mathcal{G}^-(h_2), \mathcal{G}^-(h_3), \cdots,\\
&\quad \mathcal{G}^-(h_{4k-8}), \mathcal{G}^-(h_{4k-7}), \mathcal{G}^-(h_{4k-6}),\\
&\quad \mathcal{G}^-(h_{4k-5}), \mathcal{G}^-(h_{4k-4}), \mathcal{G}^-(h_{4k-1})\}\\
&= \operatorname{mex}\{1,0,2,0,\cdots,2(k-2),2(k-2)+1,2(k-2)+2,\\
&\quad 2(k-2)+1,2(k-1),2(k-1)+1\} \text{ by induction}\\
&= \operatorname{mex}\{1,0,2,0,\cdots,2k-4,2k-3,2k-2,2k-3,2k-2,2k-1\}\\
&= 2k,\\
\mathcal{G}^-(h_{4k}+\mathit{2}) &= \operatorname{mex}\{\mathcal{G}^-(h_{4k}), \mathcal{G}^-(h_{4k})\oplus 1, \mathcal{G}^-(h_0+\mathit{2}), \mathcal{G}^-(h_1+\mathit{2}), \mathcal{G}^-(h_2+\mathit{2}),\\
&\quad \mathcal{G}^-(h_3+\mathit{2}), \cdots, \mathcal{G}^-(h_{4k-8}+\mathit{2}), \mathcal{G}^-(h_{4k-7}+\mathit{2}), \mathcal{G}^-(h_{4k-6}+\mathit{2}),\\
&\quad \mathcal{G}^-(h_{4k-5}+\mathit{2}), \mathcal{G}^-(h_{4k-4}+\mathit{2}), \mathcal{G}^-(h_{4k-1}+\mathit{2})\}\\
&\quad \text{by Proposition 2.1.2}\\
&= \operatorname{mex}\{2k, 2k\oplus 1, 2, 3, 0, 2, \cdots, 2(k-2)\oplus 2, (2(k-2)+1)\oplus 2,\\
&\quad (2(k-2)+2)\oplus 2, (2(k-2)+1)\oplus 2, (2(k-1))\oplus 2,\\
&\quad (2(k-1)+1)\oplus 2\} \text{ by induction}\\
&= \operatorname{mex}\{2k, 2k\oplus 1, 2, 3, 0, 2, \cdots, (2k-4)\oplus 2, (2k-3)\oplus 2,\\
&\quad (2k-2)\oplus 2, (2k-3)\oplus 2, (2k-2)\oplus 2, (2k-1)\oplus 2\}\\
&= 2k\oplus 2,\\
\mathcal{G}^-(h_{4k}+\mathit{2}+\mathit{2}) &= \operatorname{mex}\{\mathcal{G}^-(h_{4n}+\mathit{2}), \mathcal{G}^-(h_{4n}+\mathit{2})\oplus 1, \mathcal{G}^-(h_0+\mathit{2}+\mathit{2}),\\
&\quad \mathcal{G}^-(h_1+\mathit{2}+\mathit{2}), \mathcal{G}^-(h_2+\mathit{2}+\mathit{2}), \mathcal{G}^-(h_3+\mathit{2}+\mathit{2}), \cdots,\\
&\quad \mathcal{G}^-(h_{4k-8}+\mathit{2}+\mathit{2}), \mathcal{G}^-(h_{4k-7}+\mathit{2}+\mathit{2}), \mathcal{G}^-(h_{4k-6}+\mathit{2}+\mathit{2}),\\
&\quad \mathcal{G}^-(h_{4k-5}+\mathit{2}+\mathit{2}), \mathcal{G}^-(h_{4k-4}+\mathit{2}+\mathit{2}), \mathcal{G}^-(h_{4k-1}+\mathit{2}+\mathit{2})\}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \text{by Proposition 2.1.2} \\
= &\quad \text{mex}\{2k \oplus 2, 2k \oplus 3, 0, 1, 2, 0, \cdots, 2(k-2), 2(k-2)+1, \\
&\quad \text{by induction} \\
&\quad 2(k-2)+2, 2(k-2)+1, 2(k-1), 2(k-1)+1\} \\
= &\quad \text{mex}\{2k \oplus 2, 2k \oplus 3, 0, 1, 2, 0, 2k-4, 2k-3, 2k-2, 2k-3, \\
&\quad 2k-2, 2k-1\} \\
= &\quad 2k.
\end{aligned}
$$

Therefore, by induction $\Gamma(h_{4k}) = (2k-1)^{(2k)(2k\oplus 2)}$.

Since every heap of size four or greater of $\mathcal{N}(0.3103)$ is wild, 0.3103 is not domesticatable. ■

Proposition 3.3.8 shows something which is very common with non-domesticatable games - namely that nimifying certain games, rather than helping, actually makes things worse. In the game 0.3103, heaps of size $5n+2$ for $n \in \mathbb{N}$ have wild genus, all others have genus equal to a tame value. However, as the proof of Proposition 3.3.8 shows, after nimifying, $\mathcal{N}(0.3103)$ every heap of size greater than four is wild.

3.4 Future Work

Finite subtraction octal games are an excellent starting point for further investigation in misère game theory. They have enough structure that they are not all tame, like subtraction games, but are not too difficult to analyse. The fact that the rules are completely encoded in a string of digits means that it is easy to write computer programs to quickly determine the genera of heaps.

We conclude this chapter with the three open questions presented in this chapter regarding finite subtraction octal games, namely:

1) Are the periods of each of the "columns" of the genera of the heaps of a finite subtraction octal game related? Can we prove Conjecture 3.3.2?

2) Can we further develop the Nimming Number and possibly prove or disprove Conjecture 3.3.5?

3) Why are some finite subtraction octal games domesticatable, some almost domesticatable, and some not domesticatable?

Chapter 4

Toppling Towers

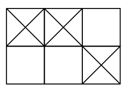
In this chapter we use the genus to analyse a non-heap based game.

4.1 The Basics

4.1.1 Rules

Definition. The impartial combinatorial game **Toppling Towers** is played as follows: a $n \times m$ board is given, along with the placement of $k \leq n \times m$ towers on the board. On her turn, a player can "topple" a tower in one of the four cardinal directions. Upon falling, the tower then also topples all contiguous towers in the direction in which it was toppled. Towers which have been toppled are then removed from the board. Under the normal play convention, a player loses if she has no towers to topple. Under the misère play convention, a player wins if she has no towers to topple.

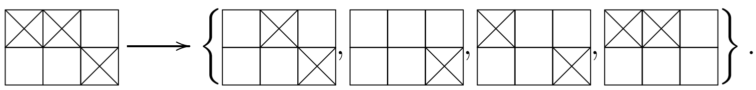
Notation. We denote a game of Toppling Towers by a grid where an $\times$ represents a tower and a blank space represents a space in which there is no tower. For example, a toppling towers game played on a 2×3 board with towers at positions $(1, 1)$, $(1, 2)$, and $(2, 3)$ is represented by



.

We will play under the convention that a tower can topple off the board.

Example 4.1.1. Returning to the example given above, the available moves are to the following positions:



Proposition 4.1.1. *Let G be a Toppling Towers game with no towers (i.e. the empty board) and let H be a Toppling Towers game with exactly one tower placed somewhere on the board. Then $\Gamma(G) = 0^{120}$ and $\Gamma(H) = 1^{031}$.*

Proof. Since there are no moves from G, by definition of genus, $\Gamma(G) = 0^{120}$. There is only one move from H, $H \longrightarrow G$, and by Proposition 2.1.2, $\Gamma(H) = 1^{031}$. ■

4.1.2 Disjunctive Gameplay

Since a tower can only effect contiguous towers, we see that a Toppling Towers game can be thought of as the disjunctive sum of each of the contiguous components, ignoring any empty squares. Played under the normal play convention, to determine the game value, all one needs to do is determine the game value of each of the components and then take their Nim sum, as per Proposition 1.3.7. Under the misère play convention, if all of the components are tame, we can compute the genus of the game by computing each of the components' genus, and summing them under the rules given in Theorem 2.4.2.

Example 4.1.2. Returning again to

$$\begin{array}{|c|c|c|}\hline \boxtimes & \boxtimes & \\ \hline & & \boxtimes \\ \hline \end{array},$$

we can think of this game as the disjunctive sum of the following two games:

$$\begin{array}{|c|c|}\hline \boxtimes & \boxtimes \\ \hline \end{array}, \begin{array}{|c|}\hline \boxtimes \\ \hline \end{array}.$$

4.2 Analysis of $1 \times m$ Boards

We begin our analysis with the easiest case.

Notation. For $n \in \mathbb{Z}^{\geq 0}$, we will take $\left(\boxtimes\right)^n$ to mean a row of n towers.

Proposition 4.2.1. *Let $G = \left(\boxtimes\right)^n$ for $n \in \mathbb{N}$. Then*

$$\Gamma(G) = \begin{cases} 1^{031} & \text{if } n = 1 \\ n^{n(n\oplus 2)} & \text{if } n > 1. \end{cases}$$

Proof. Proceed by induction on n. If $n = 1$, then this follows from Proposition 4.1.1.

Suppose $n = 2$. That is, we have the game

$$\boxtimes\boxtimes\,.$$

The available moves are to

$$\boxtimes\boxtimes \longrightarrow \{\boxtimes, \square\}\,.$$

By Propositions 2.1.2 and 4.1.1, the genus of this game is 2^{20}.

Suppose the induction hypothesis holds $\forall\ n < k$. Consider the game

$$(\boxtimes)^k\,.$$

By toppling towers to the left from the j^{th} left most tower, we obtain games with genera $i^{i(i\oplus 2)}$ for $i \in \mathbb{N}$, $2 \le i < k$, as well as games with genera 0^{120} and 1^{031} by toppling over all towers and all but one tower respectively.

Consider now toppling a middle tower to the north or south, leaving the game

$$(\boxtimes)^a\square(\boxtimes)^b,$$

for $a + 1 + b = k$. Since $a, b < k$, by induction, we know their genera, and thinking of this position as the disjunctive sum of $(\boxtimes)^a$ and $(\boxtimes)^b$, by Theorem 2.4.2, the genus of this position is $(a \oplus b)^{(a\oplus b)(a\oplus b\oplus 2)}$. By Proposition 1.3.6, $a \oplus b \le a + b < k$. Thus, there does not exist a position of this game with genus $k^{k(k\oplus 2)}$.

By Proposition 2.1.2, $\Gamma(G) = k^{k(k\oplus 2)}$, as required. ■

Corollary 4.2.2. *Let*

$$G = (\boxtimes)^{i_1} + (\boxtimes)^{i_2} + \cdots + (\boxtimes)^{i_n} + \sum_{k=1}^{m} \boxtimes\,,$$

where $i_j \ge 2$ *for* $j \in \{1, 2, \cdots, n\}$. *Let*

$$\upsilon = \begin{cases} 0 & \textit{if } m \equiv 0 \textit{ mod (2)} \\ 1 & \textit{if } m \equiv 1 \textit{ mod (2).} \end{cases}$$

Then

$$G \in \mathcal{P} \textit{ under the misère play convention} \iff i_1 \oplus i_2 \oplus \cdots \oplus i_n \oplus \upsilon = 0.$$

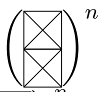
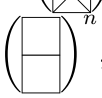
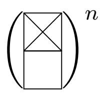
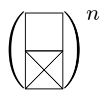
Proof. Recall from Proposition 2.3.1. $G \in \mathcal{P}$ under the misère play convention $\iff$ the first superscript in the genus symbol of G equals 0. Since each of the summands in the game disjunctive sum is tame, by Theorem 2.4.2,

$$\Gamma(G) = i_1 \oplus i_2 \oplus \cdots \oplus i_n \oplus v^{(i_1 \oplus i_2 \oplus \cdots \oplus i_n \oplus v)(i_1 \oplus i_2 \oplus \cdots \oplus i_n \oplus v \oplus 2)},$$

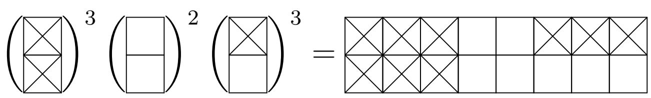
which gives the result. ■

4.3 Analysis of $2 \times m$ Boards

For a complete list of the genera of all game positions with eight or fewer towers on $2 \times m$ boards, refer to Appendix B.

Notation. For $n \in \mathbb{Z}^{\geq 0}$, we take  to mean two rows of n towers stacked on top of each other. Similarly, for , , and .

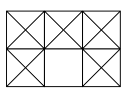
Example 4.3.1.



4.3.1 Wildness in Toppling Towers

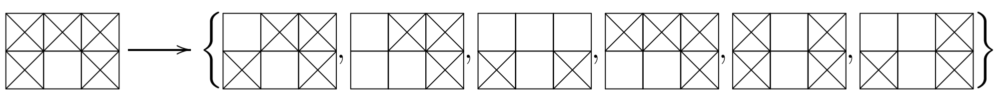
Unlike in the $1 \times m$ case, where all positions are tame, we are now able to discover wild Toppling Towers positions, the simplest occurring with five towers.

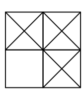
Proposition 4.3.1. *The Toppling Towers position*

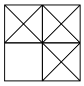


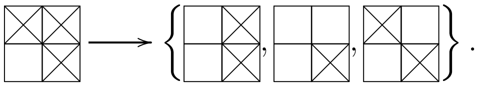
is wild.

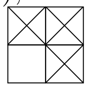
Proof. The available moves, up to symmetry, are

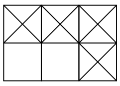


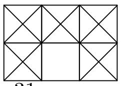
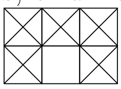
Claim that  is tame.

The available moves, up to symmetry, from  are

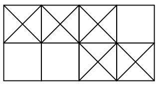
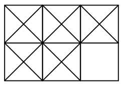


which have genera 2^{20} (Proposition 4.2.1) , 1^{031} (Proposition 4.1.1), and 0^{120} (Proposition 4.1.1 and Theorem 2.4.2) respectively. By Proposition 2.1.2,  is tame with genus 3^{31}.

Similarly,  is tame.

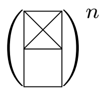
Then all options of  are tame and calculations give that their genera are 2^{20}, 3^{31}, 0^{120}, 4^{46}, 0^{02}, 3^{31} respectively. By Theorem 2.2.2, the game is wild since it has both 0^{120} and 0^{02} as options, and no 1^{031} or 1^{13} as options. Equally, calculating directly gives that the genus of  is 1^{531}, which is wild. ∎

This is hardly an isolated case. There are two other positions with five towers which are wild,

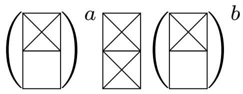
 , and  ,

as well as numerous positions with more than five towers which are wild - see Appendix B for some examples.

4.3.2 Some Tame Positions

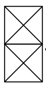
Clearly, from Proposition 4.2.1, all $2 \times m$ positions of the form  are tame. We are also able to show that following positions are tame:

Theorem 4.3.2. *Let*

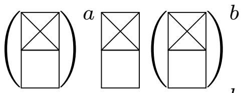
$G =$  ,

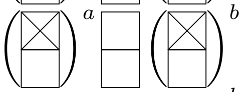
for $a, b \in \mathbb{Z}^{\geq 0}$. *Then* G *is tame.*

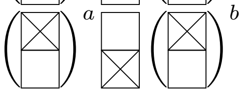
Proof. Let $n = a + b + 1$. Proceed by induction on n.

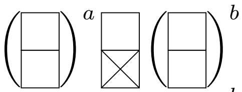
When $n = 1$, $G =$ . By Proposition 4.2.1, $\Gamma(G) = 2^{20}$, which is tame. This shows the base case.

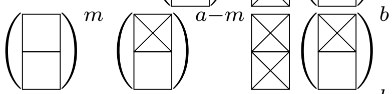
Suppose true $\forall\ n < k$. Consider $n = k$. That is, $a + b + 1 = k$. The available moves, up to "symmetry", are to positions of the following form:

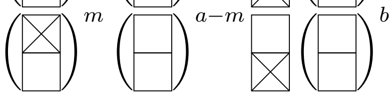
 Tame by Proposition 4.2.1,

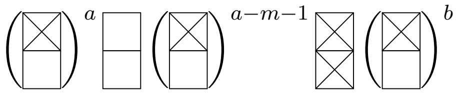
 Tame by Proposition 4.2.1 and Theorem 2.4.2,

 Tame by Propositions 4.1.1 and 4.2.1, and Theorem 2.4.2,

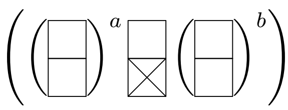
 Tame by Proposition 4.1.1,

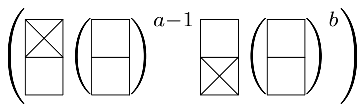
 Tame by induction,

 Tame by Propositions 4.1.1 and 4.2.1, and Theorem 2.4.2,

 Tame by induction, Proposition 4.2.1, and Theorem 2.4.2.

Note that

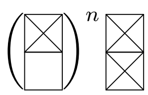
Γ  $= 1^{031}$ by Proposition 4.1.1,

Γ  $= 0^{120}$ by Theorem 2.4.2,

both of which are options of G. By Theorem 2.2.2, G is tame. ■

In fact, for specific configurations of the above form, we can say more than their genera being tame, we have a formula to determine their genera:

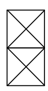
Proposition 4.3.3. *Let*

$G =$  ,

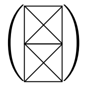
for $n \in \mathbb{Z}^{\geq 0}$. *Then* $\Gamma(G) = (n+2)^{(n+2)((n+2)\oplus 2)}$.

Proof. Proceed by induction on n.

Suppose $n = 0$. Then we have the game

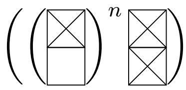
 .

By Proposition 4.2.1,

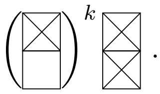
Γ  $= 2^{20} = (0+2)^{(0+2)((0+2)\oplus 2)}.$

which shows the base case.

Suppose true for $n < k$. That is

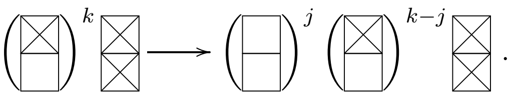
Γ  $= (n+2)^{(n+2)((n+2)\oplus 2)}.$

Consider



By Theorem 4.3.2, there exists moves to positions with genera 0^{120} and 1^{031}, namely toppling (1,2) right and (1,1) right respectively.

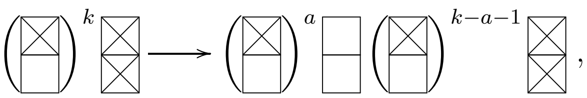
Now for $1 \leq j \leq k$, let G' be the game obtained by toppling $(1, j)$ to the left.



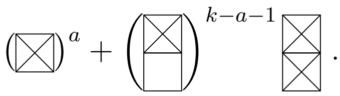
By induction, $\Gamma(G') = (k-j+2)^{(k-j+2)((k-j+2)\oplus 2)}$. As j ranges from 1 to k, we get options with genera of the form $m^{m(m\oplus 2)}$ for all m such that $2 \leq m \leq k+1$.

Thus, we have options of G with genera 0^{120}, 1^{031}, and $m^{m(m\oplus 2)}$ for $2 \leq m \leq k+1$. If we can show that there are no other moves to options H with $\Gamma(H) = (k+2)^{(k+2)(k+2)\oplus 2}$, then by Proposition 2.1.2, $\Gamma(G) = (k+2)^{(k+2)(k+2)\oplus 2}$.

Consider toppling one of the towers in the top row either up or down:



for $a \in \mathbb{N}$, $1 \leq a \leq k-1$. That is, we have the position



By Proposition 4.2.1,

Γ  $= \begin{cases} 1^{031} & \text{if } a = 1 \\ a^{a(a\oplus 2)} & \text{else.} \end{cases}$

By induction

$$\Gamma\left(\left(\begin{smallmatrix}\boxtimes\\\square\end{smallmatrix}\right)^{k-a-1}\begin{smallmatrix}\boxtimes\\\boxtimes\end{smallmatrix}\right)=(k-a+1)^{(k-a+1)((k-a+1)\oplus 2)}.$$

By Theorem 2.4.2,

$$\Gamma\left(\left(\boxtimes\right)^{a}+\left(\begin{smallmatrix}\boxtimes\\\square\end{smallmatrix}\right)^{k-a-1}\begin{smallmatrix}\boxtimes\\\boxtimes\end{smallmatrix}\right)=(a\oplus(k-a+1))^{(a\oplus(k-a+1))((a\oplus(k-a+1))\oplus 2)}.$$

By Proposition 1.3.6

$$a\oplus(k-a+1)\leq a+k-a+1=k+1<k+2.$$

Consider now toppling one of the towers in the top row to the right.

$$\left(\begin{smallmatrix}\boxtimes\\\square\end{smallmatrix}\right)^{k}\begin{smallmatrix}\boxtimes\\\boxtimes\end{smallmatrix}\longrightarrow\left(\begin{smallmatrix}\boxtimes\\\square\end{smallmatrix}\right)^{k-b}\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)^{b}\begin{smallmatrix}\square\\\boxtimes\end{smallmatrix},$$

for $b\in\mathbb{Z}^{\geq 0}$, $0\leq b\leq k$. That is, we have the game

$$\left(\boxtimes\right)^{k-b}+\boxtimes.$$

By Proposition 4.2.1,

$$\Gamma\left(\left(\boxtimes\right)^{k-b}\right)=\begin{cases}1^{031} & \text{if } b=k\\ (k-b)^{(k-b)((k-b)\oplus 2)} & \text{else,}\end{cases}$$

and

$$\Gamma\left(\boxtimes\right)=1^{031}.$$

By Theorem 2.4.2

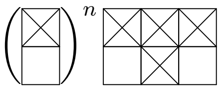
$$\left(\boxtimes\right)^{k-b}+\boxtimes=\begin{cases}1^{031} & \text{if } b=k\\ 0^{120} & \text{if } b=k-1\\ ((k-b)\oplus 1)^{((k-b)\oplus 1)((k-b)\oplus 3)} & \text{else.}\end{cases}$$

By Proposition 1.3.6

$$k-b\oplus 1\leq k-b+1<k+2.$$

Since no other options of G have genus equal to $(k+2)^{(k+2)(k+2)\oplus 2}$, $\Gamma(G)=(k+2)^{(k+2)(k+2)\oplus 2}$. ■

Proposition 4.3.4. *Let*

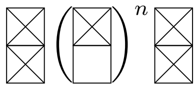
$G =$  ,

for $n \in \mathbb{Z}^{\geq 0}$. *Then* $\Gamma(G) = ((n+3) \oplus 1)^{((n+3)\oplus 1)((n+3)\oplus 3)}$.

Proof. The method of this proof is similar to that of Proposition 4.3.3. ∎

We conclude this section with another set of tame games:

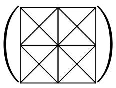
Theorem 4.3.5. *Let*

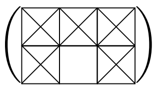
$G =$  ,

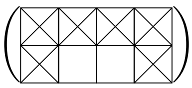
for $n \in \mathbb{Z}^{\geq 0}$. *Then* G *is tame* $\iff$ $n \equiv 0$ *(mod 4).*

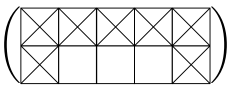
Proof. Proceed by induction on n.

Calculations give

Γ  $= 0^{02}$,

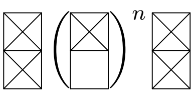
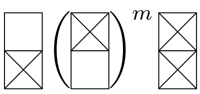
Γ  $= 1^{531}$,

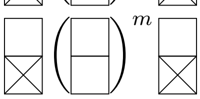
Γ  $= 6^{046}$,

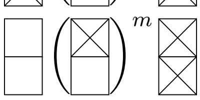
Γ  $= 1^{731}$,

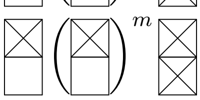
which shows the base case.

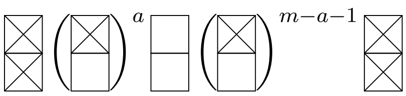
Suppose true $\forall\ n < m$. Consider $n = m$. The available moves up to symmetry are

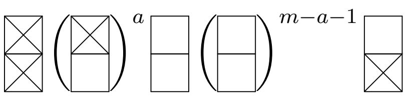
 $\longrightarrow$  $= A$,

 $= B$,

 $= C$,

 $= D$,

 $= E$ for $0 \leq a \leq m-1$,

 $= F$ for $0 \leq a \leq m-1$.

By Proposition 4.1.1, Theorem 4.3.2, and Theorem 2.4.2, all the above positions are tame. Moreover, $\Gamma(B) = 0^{120}$. By Theorem 2.2.2, if there is amongst the options of G an option with genus equal to one but not both of 0^{02} or 1^{13}, and no option equal to 1^{031}, then G is wild. Otherwise, it is tame.

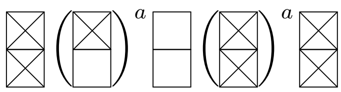
Consider the genera of the options of G:

$$\begin{aligned}
\Gamma(A) &= ((m+2)\oplus 1)^{((m+2)\oplus 1))((m+2)\oplus 3)} \text{ by Proposition 4.1.1, Proposition 4.3.3,} \\
&\qquad \text{and Theorem 2.4.2,} \\
\Gamma(B) &= 0^{120}, \\
\Gamma(C) &= (m+2)^{(m+2)((m+2)\oplus 2)} \text{ by Proposition 4.3.3,} \\
\Gamma(D) &= (m+3)^{(m+3)((m+3)\oplus 2)} \text{ by Proposition 4.3.3,} \\
\Gamma(E) &= ((a+2)\oplus(m-a+1))^{((a+2)\oplus(m-a+1))(((a+2)\oplus(m-a+1))\oplus 2)} \\
&\qquad \text{by Proposition 4.3.3 and Theorem 2.4.2,} \\
\Gamma(F) &= ((a+2)\oplus 1)^{((a+2)\oplus 1)((a+2)\oplus 3)} \text{ by Proposition 4.1.1, Proposition 4.3.3,} \\
&\qquad \text{and Theorem 2.4.2.}
\end{aligned}$$

None of the genera for A, C, D, E, or F can equal 1^{031}, since the base and the first superscript in the genus are always equal. Since $m \geq 0$, $m+2, m+3 \geq 2$, so the genera of A, C, and D are not equal to 0^{02} or 1^{13}. Since $a \geq 0$, $(a+2)\oplus 1 \geq 2$, so $\Gamma(F) \neq 0^{02}$ or 1^{13}. Thus, it remains to examine $\Gamma(E)$.

Take $m = 4k$ for $k \in \mathbb{N}$. Suppose $\Gamma(E) = 0^{02}$. Then

$$\begin{aligned}
\Gamma(E) = 0^{02} &\implies (a+2)\oplus(m-a+1) = 0 \\
&\implies a+2 = m-a+1 \\
&\implies a = m-a-1 \\
&\implies E =
\end{aligned}$$



$$\begin{aligned}
&\implies 2a+1 = 4k \\
&\implies k \notin \mathbb{Z}^{\geq 0},
\end{aligned}$$

which is a contradiction. Therefore, for $m = 4k$, $\Gamma(E) \neq 0^{02}$.

Suppose $\Gamma(E) = 1^{13}$. Then

$$\Gamma(E) = 1^{13} \implies (a+2)\oplus(m-a+1) = 1$$

$$
\begin{aligned}
&\implies |(a+2)-(m-a+1)| = 1 \\
&\implies \text{either } a+3 = m-a+1 \text{ or } a+2 = m-a+2.
\end{aligned}
$$

Suppose $a+3 = m-a+1$.

$$
\begin{aligned}
a+3 = m-a+1 &\implies a+1 = m-a-1 \\
&\implies E = \begin{array}{c}\boxtimes\\\boxtimes\end{array}\left(\begin{array}{c}\boxtimes\\\square\end{array}\right)^{a}\begin{array}{c}\square\\\square\end{array}\left(\begin{array}{c}\boxtimes\\\square\end{array}\right)^{a+1}\begin{array}{c}\boxtimes\\\boxtimes\end{array} \\
&\implies 2a+2 = 4k \\
&\implies k \notin \mathbb{Z}^{\geq 0},
\end{aligned}
$$

which is a contradiction.

Suppose $a+2 = m-a+2$.

$$
\begin{aligned}
a+2 = m-a+2 &\implies 2a = m \\
&\implies 2a = 4k \\
&\implies a = 2k \\
&\implies a+2 = 2k+2 \text{ and } m-a+1 = 2k+1 \\
&\implies (a+2) \oplus (m-a+1) = 3,
\end{aligned}
$$

which is a contradiction, as we supposed that $(a+2) \oplus (m-a+1) = 1$.

Therefore, for $m = 4k$, $\Gamma(E) \neq 1^{13}$. By Theorem 2.2.2, for $m = 4k$, G is tame.

Take $m = 4k+1$. Consider the move

$$
G \longrightarrow \begin{array}{c}\boxtimes\\\boxtimes\end{array}\left(\begin{array}{c}\boxtimes\\\square\end{array}\right)^{2k}\begin{array}{c}\square\\\square\end{array}\left(\begin{array}{c}\boxtimes\\\square\end{array}\right)^{2k}\begin{array}{c}\boxtimes\\\boxtimes\end{array} = E'.
$$

By Theorem 2.4.2, $\Gamma(E') = 0^{02}$. By Theorem 2.2.2, for $m = 4k+1$, G is wild.

Take $m = 4k+2$. Consider the move

$$
G \longrightarrow \begin{array}{c}\boxtimes\\\boxtimes\end{array}\left(\begin{array}{c}\boxtimes\\\square\end{array}\right)^{2k}\begin{array}{c}\square\\\square\end{array}\left(\begin{array}{c}\boxtimes\\\square\end{array}\right)^{2k+1}\begin{array}{c}\boxtimes\\\boxtimes\end{array} = E''.
$$

By Theorem 2.4.2, $\Gamma(E'') = 1^{13}$. By Theorem 2.2.2, for $m = 4k+2$, G is wild.

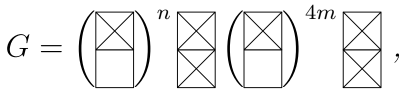
Take $m = 4k+3$. Consider the move

$$
G \longrightarrow \begin{array}{c}\boxtimes\\\boxtimes\end{array}\left(\begin{array}{c}\boxtimes\\\square\end{array}\right)^{2k+1}\begin{array}{c}\square\\\square\end{array}\left(\begin{array}{c}\boxtimes\\\square\end{array}\right)^{2k+1}\begin{array}{c}\boxtimes\\\boxtimes\end{array} = E'''.
$$

By Theorem 2.4.2, $\Gamma(E''') = 0^{02}$. By Theorem 2.2.2, for $m = 4k+3$, G is wild.

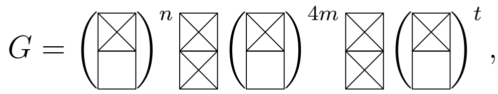
This completes the induction. $\blacksquare$

Corollary 4.3.6. *Let*



for $n, m \in \mathbb{Z}^{\geq 0}$. Then G is tame.

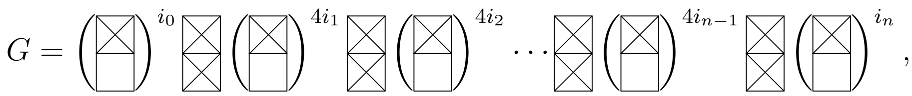
Corollary 4.3.7. *Let*



for $n, m, t \in \mathbb{Z}^{\geq 0}$. Then G is tame.

The following is the most important corollary of Theorem 4.3.5:

Corollary 4.3.8. *Let*

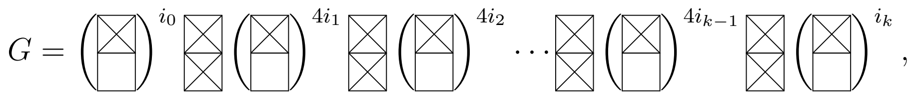


for $i_0, i_1, \cdots, i_n \in \mathbb{Z}^{\geq 0}$ and such that if $i_j = 0$, then $i_{j+1} \neq 0$ for $j \in \{1, 2, \cdots, n-2\}$. Then G is tame.

Proof. Let u denote the number of towers in the bottom row. Proceed by induction on u.

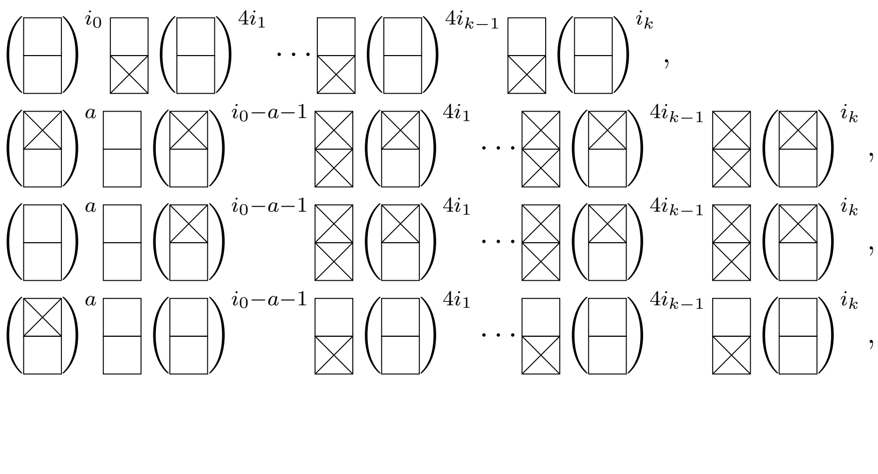
If $u = 1$, then by Theorem 4.3.2, G is tame.

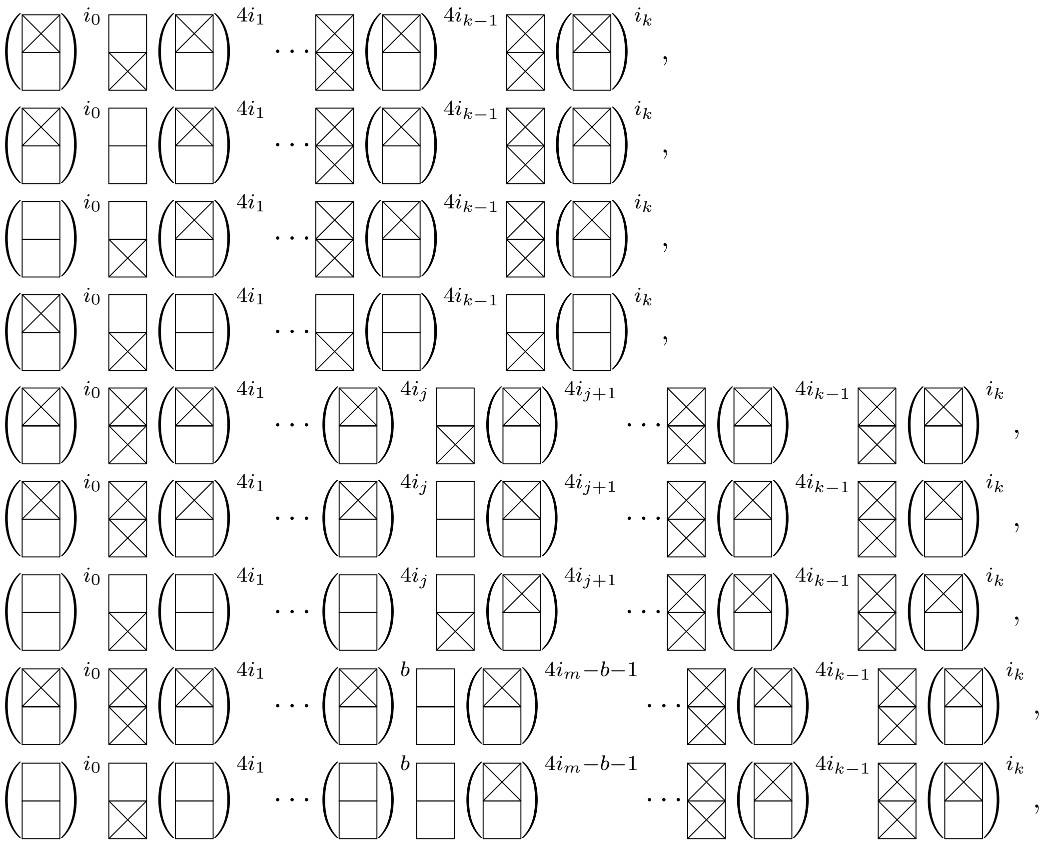
Suppose now true for $u < k$. Consider $u = k$.



such that if $i_j = 0$, then $i_{j+1} \neq 0$ for $j \in \{1, 2, \cdots, k-2\}$.

The available moves are to positions of the following forms:

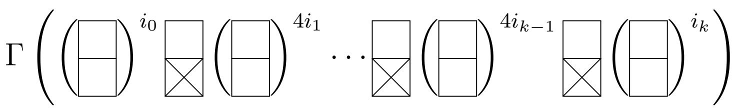




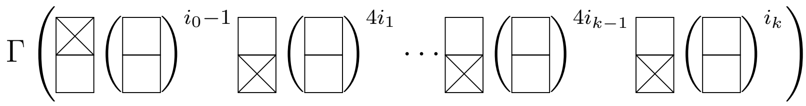
for

$$\begin{aligned} 0 \leq \quad & a \quad \leq i_0 - 1, \\ 1 \leq \quad & j \quad \leq k - 1, \\ 1 \leq \quad & m \quad \leq k - 1, \\ 0 \leq \quad & b \quad \leq 4i_m - 1. \end{aligned}$$

By induction, Proposition 4.2.1, and Theorem 2.4.2, all of the positions are tame. If $k \equiv 0 \pmod 2$, by Proposition 4.1.1 and Theorem 2.4.2,

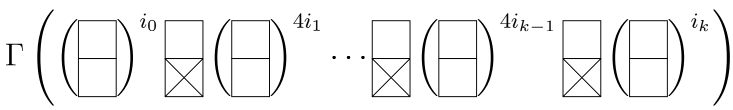


$$= \quad 0^{120},$$

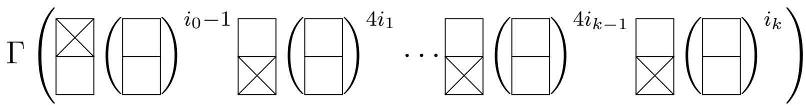


$$= \quad 1^{031}.$$

If $k \equiv 1 \pmod 2$, by Proposition 4.1.1 and Theorem 2.4.2,



$$= \quad 1^{031},$$



$= 0^{120}.$

By Theorem 2.2.2, G is tame. ■

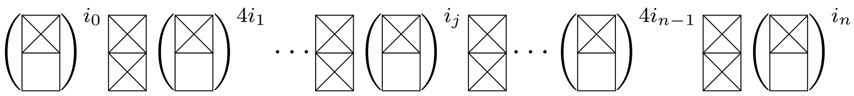
4.3.3 All the Tame and Wild Positions for $2 \times m$ Boards

Proposition 4.3.1 showed us that there exist wild positions in $2\times m$ boards. Theorem 4.3.2 and Corollary 4.3.8 gave us a set of tame positions. In fact, these are the only contiguous tame positions of Toppling Towers in $2 \times m$ boards with towers in both rows; all other contiguous positions are wild.

Theorem 4.3.9. *The only contiguous tame positions in $2 \times m$ boards with towers in both rows are those of the form given in Theorem 4.3.2 and Corollary 4.3.8.*

Proof. Suppose we have a position not given in Theorem 4.3.2 and Corollary 4.3.8. There must be at least two towers in the bottom row to not be a position given in Theorem 4.3.2. To not be a position in Corollary 4.3.8, one of the following two conditions must be satisfied:

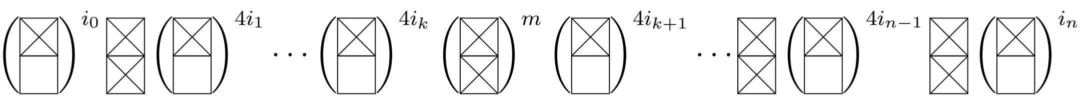
1)



,

such that $1 \leq j \leq n-1$, $i_j \not\equiv 0 \pmod 4$.

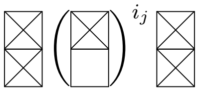
2)



,

such that $1 \leq k \leq n-1$, $m \geq 3$.

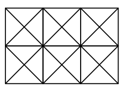
Suppose we are in Case 1). Then one of the followers of the position is



,

which is wild by Theorem 4.3.5. Therefore the position given in Case 1) is wild.

Suppose we are in Case 2). Then one of the followers of the position is



,

which has genus 2^{1520}. Therefore the position given in Case 2) is wild.

Therefore the only contiguous tame positions in $2 \times m$ boards are those given in Theorem 4.3.2 and Corollary 4.3.8. ■

4.4 Future Work

Toppling Towers provides an excellent starting point for impartial misère games which are not taking and breaking games. As with taking and breaking games, there is enough structure that these games are not misère Nim in disguise.

The following areas need to be examined:

1) Writing a program to determine the genera of positions in Toppling Towers.

2) Continue the work of Propositions 4.3.3 and 4.3.4 to find formulae for the genera of all the contiguous tame positions on $2 \times m$ boards.

3) Analyse larger board positions.

Chapter 5

Indistinguishablility

One of the main problems with using genus as a tool to classify impartial misère games is the inability to easily determine the genus of the disjunctive sum of two wild games or even, in most cases, the disjunctive sum of a wild game with a tame one. Recently Thane Plambeck has developed a tool that can be used to determine the outcome class of disjunctive sums of wild and tame positions within the same game, which he calls the **indistinguishablility quotient**. Further to this, Aaron Siegel has developed the software package *misère solver* which allows us to easily apply Plambeck's method to misère finite octal games. The role of this chapter is to provide a brief overview of the indistinguishablility quotient; formal proofs are left to [8]. In Section 5.4.4, we provide an example of Plambeck's method analysing 0.3103.

5.1 The Alphabet

For each $n \in \mathbb{N}$, define a formal symbol h_n. Let $\mathcal{H} = \{h_1, h_2, h_3, \cdots, \}$. We call $\mathcal{H}$ the **heap alphabet**. Define $\mathcal{H}_n = \{h_1, h_2, \cdots, h_n\}$. Let $\mathcal{F}_{\mathcal{H}}$ denote the free Abelian monoid on the heap alphabet $\mathcal{H}$, where we denote the binary operation multiplicatively by $\cdot$ and the identity is the empty string, denoted by 1.

Suppose we are given a taking and breaking game G with heaps denoted by $\hat{h_0}, \hat{h_1}, \hat{h_2}, \cdots$. Then there is a clear correspondence between elements of $\mathcal{F}_{\mathcal{H}}$ and the positions in G. For example, suppose in G we have the position a heap of size two, a heap of size three, and two heaps of size six; that is, $\hat{h_2} + \hat{h_3} + \hat{h_6} + \hat{h_6}$. We can easily see that this corresponds to the element $h_2 \cdot h_3 \cdot {h_6}^2$ in $\mathcal{F}_{\mathcal{H}}$, and, due to the multiplicative structure of $\mathcal{F}_{\mathcal{H}}$, we denote a heap of size zero, $\hat{h_0}$, by the empty string, 1. In an abuse of notation, we generally denote both heaps in the game G and letters in $\mathcal{F}_{\mathcal{H}}$ by h_n.

Definition. Let $\mathcal{F} : \mathcal{H} \longrightarrow \mathcal{F}_{\mathcal{H}}$ be the free functor on the heaps of $\mathcal{H}$. That is, $\mathcal{F}$

takes h_n, a heap of size n in the game G, to the letter with which it is associated in $\mathcal{F}_{\mathcal{H}}$. We extend $\mathcal{F}$ homomorphically over the disjunctive sum in $\mathcal{H}$.

For the rest of this chapter, we will associate game positions with their words under $\mathcal{F}_{\mathcal{H}}$.

5.2 Indistinguishablility

Fix an impartial taking and breaking game G, as well as the play convention (either normal or misère), and let $\mathcal{F}_{\mathcal{H}}$ be the free Abelian monoid on the heap alphabet of G. Take $u, v \in \mathcal{F}_{\mathcal{H}}$.

Definition. For u, v as defined above, we say that u **is indistinguishable from** v **over** $\mathcal{F}_{\mathcal{H}}$ if for every element $w \in \mathcal{F}_{\mathcal{H}}$, $u \cdot w$ and $v \cdot w$ are in the same outcome class taken as positions in G.

Notation. If u is indistinguishable from v, we write $u \; \rho \; v$.

Lemma 1 of [8] shows that ρ is an equivalence relation on $\mathcal{F}_{\mathcal{H}}$. Lemma 1 also shows that ρ is compatible with the operation "$\cdot$".

Recall from Section 1.1.7, the definition of normal play convention game equivalence: $G = H$ if for all X, $G + X$ has the same outcome as $H + X$. We can see that there are similarities between these equivalence and indistinguishablility, however the two are not interchangeable. In normal play convention game equivalence, G, H, and X, do not need to be played under the same rule set (i.e. alphabet), whereas u, v, and w are all taken from the same $\mathcal{F}_{\mathcal{H}}$. In normal play convention game equivalence, we are looking for equivalence over all games, whereas indistinguishablility concerns itself only with positions in the fixed game G. Two games may be indistinguishable under the game alphabet, but a game outside the alphabet might distinguish them.

5.3 The Indistinguishablility Quotient

Definition. Taking a fixed game G, with ρ the indistinguishablility relation on $\mathcal{F}_{\mathcal{H}}$, for $\mathcal{H}$ the heap alphabet of G, we define the **indistinguishablility quotient of G**

or **quotient monoid**, denoted by $\mathcal{Q}(G)$, as follows

$$\mathcal{Q}(G) = \mathcal{F}_{\mathcal{H}}/\rho.$$

Suppose we have u, v in the same congruence class of $\mathcal{Q}(G)$. Then $u \,\rho\, v$. That is, for every game w in $\mathcal{F}_{\mathcal{H}}$, $u \cdot w$ and $v \cdot w$ are in the same outcome class. Taking $w = 1$, we see that u and v are in the same outcome class. Thus all elements of a particular congruence class are in the same outcome class.

A monoid structure can be placed on $\mathcal{Q}(G)$ as follows:

- Define the binary operation $\cdot_{\mathcal{Q}(G)}$ on $\mathcal{Q}(G)$ as follows: $[u]_\rho \cdot_{\mathcal{Q}(G)} [v]_\rho = [u \cdot v]_\rho$.

- Define the identity $[e]_\rho$ to be the set of positions under the alphabet of G which are indistinguishable from the empty game.

Suppose we have a game G played under the normal play convention, and consider $\mathcal{Q}(G)$. Consider a position u in G. The Sprague-Grundy Theory says that u is equivalent to some Nim heap, say $\Bbbk$. Then $[u]_\rho \in \mathcal{Q}(G)$ is the congruence class of all positions using the alphabet of G which are also equivalent to $\Bbbk$, so we can think of $\mathcal{Q}(G)$ as a set of Nim positions, and, since $*_k + *_k = 0$, under the normal play convention, $\mathcal{Q}(G)$ is a group with each element self-inverse. However, for games played under the misère game convention, $\mathcal{Q}(G)$ does not necessarily have inverses, and the most that can be said is that $\mathcal{Q}(G)$ is a monoid.

Definition. Given a set X, a **partitioning function** is a map $\varphi : X \longrightarrow \{P, N\}$.

Example 5.3.1. For a game G, the function which assigns to each position of G its outcome class is a partioning function.

5.4 Finite Octal Games

[8] presents the following method for analysing certain misère finite octal games:

1) Fix an $n \in \mathbb{N}$.

2) Considering only heaps of size n or less (i.e. $\mathcal{F}_{\mathcal{H}_n}$), find its "indistinguishablility quotient" $\mathcal{M}_n$. Although any Abelian monoid which satisfies conditions 3) and 5) and can be proved correct as per 6) will work, we will take $\mathcal{M}_n \cong \mathcal{F}_{\mathcal{H}_n}/\rho$. The trick is determining all the relations on $\mathcal{F}_{\mathcal{H}_n}/\rho$.

3) Define a **pretending function** $\Phi_n : \mathcal{H}_n \longrightarrow \mathcal{M}_n$ which takes each heap of size n or less to an element of $\mathcal{M}_n$. For $\mathcal{M}_n \cong \mathcal{F}_{\mathcal{H}_n}/\rho$, we take Φ_n as $\Phi_n(h_m) = [\mathcal{F}(h_m)]_\rho$. Extend Φ_n homomorphically over $\mathcal{F}_{\mathcal{H}_n}$. The generators are of $\mathcal{M}_n$ are $\Phi_n(h_1), \Phi_n(h_2), \cdots, \Phi_n(h_n)$.

4) Using the Knuth-Bendix rewriting process ([7]), find a canonical presentation for $\mathcal{M}_n$ ([2]). That is, we can write $\mathcal{M}_n$ as a list of elements and relations, such that the product of any number of elements reduced under the rules of the relations yields an element in the list, and for all other such lists of elements and relations for $\mathcal{M}_n$, the two are isomorphic.

5) Apply a partioning function to the elements of $\mathcal{M}_n$. That is, for each element of the monoid, assign whether it is in $\mathcal{P}$ or in $\mathcal{N}$.

6) Prove that the analysis is correct. That is, show that the following diagram commutes

$$\begin{tikzcd} \mathcal{M}_n \arrow[dr, "\varphi"] & \\ \mathcal{H} \arrow[u, "\Phi_n"] \arrow[r, "\varphi"'] & \{\mathcal{P}, \mathcal{N}\} \end{tikzcd}$$

with the appropriate restrictions to heaps of size n or less. That is, Φ and the partioning functions describe all the Previous and Next positions, provided all moves are on heaps of size n or less. Not surprisingly, we call the partioning of the monoid into its "outcome classes" the **outcome partition of** $\mathcal{M}_n$. For an element $u \in \mathcal{M}_n$, we look to see the simplest position which could generate u, say h with $\Phi(h) = u$. If $\varphi(h) \in \mathcal{O}$, then $\varphi(u)$ must also be in outcome class $\mathcal{O}$ for the diagram to commute. This is how we determine the outcome partition. Once an outcome partition has been determined, we then check that the diagram commutes for all positions.

7) Repeat for $m > n$.

8) Should there exist a point where the following occurs:

   For λ the length of the octal, if there exists $p, r_0 \in \mathbb{Z}^{\geq 0}$ such that $\Phi(h_{r+p}) = \Phi(h_r)$ for all r such that $r_0 \leq r < 2r_0 + p + \lambda$, and the analysis is correct (as per 6)) for all r such that $r_0 \leq r < 2r_0 + p + \lambda$,

   then this monoid is correct to heap size r for all $r \geq r_0$. [8] presents a proof of this statement in Section 10. We have then found a monoid which completely describes the behaviour of G and whose elements are periodic with period p. That is, if we reach a point at which by increasing the heap size considered, we are not obtaining any new elements or relations in our monoid and the elements we are obtaining have become periodic in nature and the analysis is correct, then the monoid associated with G is periodic after that point and we can analyse the misère octal game through the rules of the monoid with which it is associated. Call this monoid $\mathcal{M}$ and the pretending function obtained from extending Φ_n over all heaps Φ.

9) Once we have shown that $\mathcal{M}$ correctly describes the structure of the game G (or at any of the earlier steps with $\mathcal{M}_n$ and restricting G to heaps of size n or less), given a position in G, say h, pull this position through into the monoid using $\Phi \circ \mathcal{F}$ and reduce it under the relations of the monoid to an element u in the canonical form of $\mathcal{M}$. Through the outcome partition, we can determine the outcome class of u. Not only that, but under the rules of G, we can replace h by any other element h' such that $\Phi \circ \mathcal{F}(h') = u$, since they behave the same in the monoid.

Although steps 1) to 7) could apply to almost any sort of well-defined recursive misère game, the theorem used in 8) is only for misère finite octal games.

5.4.1 Canonical Form

In 2) in the above list, we stated that we could write $\mathcal{M}_n$ as a list of elements and relations in a canonical way, using the Knuth-Bendix rewriting process ([7]). However, the Knuth-Bendix rewriting process assumes that $\mathcal{M}_n$ is a finitely presentatable

Abelian monoid. If we work with $\mathcal{M}_n \cong \mathcal{F}_{\mathcal{H}_n}/\rho$, we can take the Abelian structure from $\mathcal{F}_{\mathcal{H}_n}$, since it doesn't matter in what order the heaps are presented. Moreover, since we are examining games from which we only have a finite number of heaps, that is, games of the form $h_{k_1} + h_{k_2} + \cdots + h_{k_i}$, then

$$\begin{aligned}\Phi \circ \mathcal{F}(h_{k_1} + h_{k_2} + \cdots + h_{k_i}) &= \Phi(h_{k_1} \cdot h_{k_2} \cdots h_{k_i}) \\ &= \Phi(h_{k_1}) \cdot \Phi(h_{k_2}) \cdots \Phi(h_{k_i}).\end{aligned}$$

Moreover, $\Phi \circ \mathcal{F}$ is onto $\mathcal{F}_{\mathcal{H}_n}/\rho$, so every element of $\mathcal{F}_{\mathcal{H}_n}/\rho$ can be written as the product of finitely many generators of $\mathcal{F}_{\mathcal{H}_n}/\rho$. By [9], $\mathcal{M}_n$ is finitely presentable.

5.4.2 Correctness

Definition. Suppose we have a position a in a misère finite octal game and a move $a \longrightarrow b$ legal under the rules of the game. Call the relation $(\Phi(a), \Phi(b))$ a **move relation in $\mathcal{M}_n$.**

Definition. A **move in $\mathcal{M}_n$** is a relation $(\Phi(a), \Phi(b))$ where the relation is defined if $a \longrightarrow b$ is a valid move in G.

For our monoid, pretending function, and partioning function to be correct, we need to show the following:

1) Given an element $u \in \mathcal{M}_n$, such that $\varphi(u) \in \mathcal{P}$, there is no move from u to another position we claim to be in $\mathcal{P}$.

2) Given an element $v \in \mathcal{M}_n$ such that v did not come from an position which has no moves and such that $\varphi(v) \in \mathcal{N}$, v has a move to a position we claim to be in $\mathcal{P}$.

To do so, we proceed as follows: Given a heap of size f, for $f \leq n$, we consider replacing h_f by various smaller heaps according to the rules of G.

$$h_f \longrightarrow \prod_t h_t.$$

In an octal game, we replace h_f by either one or two smaller heaps. We denote the associated move relation by (s_f, s).

Definition. Each move relation that is formed as above, i.e. by taking a heap of size less than or equal to n and one of its legal options, is called **move pair to heap size** n.

Note that the *to* in the nomenclature does not refer that we are moving from a heap of size m to a heap of size n, rather that these are the moves associated to heap sizes up to and including a heap of size n. Let MP_n be the set of all move pairs to heap size n.

Suppose now we are given a game with a variety of disjunctive components $G_1 + G_2 + \cdots + G_n$ in which to play. Under disjunctive game play, the player picks a component, say without loss of generality G_1, and moves in that component. That is, we have

$$G_1 + G_2 + \cdots + G_n \longrightarrow G_1' + G_2 + \cdots + G_n.$$

Then the game $G_2 + \cdots + G_n$ stays unchanged, while the only game that changes is G_1. So we can think of this game as $G_1 + \mathbf{G}$, where $\mathbf{G}$ consists of all the components in which we do not move on a given turn. We will use this idea to further analyse move pairs to heap size n. By the canonical form construction of 2), every position in the game G restricted to heaps of size n or less can be reduced to one of the canonical elements of $\mathcal{M}_n$ through the use of the relations on $\mathcal{M}_n$, say $u \in \mathcal{M}_n$. Thus if we have a position in the game G restricted to heaps of size n or less in which the next move is in the heap of size f, then we have as a move relation $(u \cdot s_f, u \cdot s)$.

Definition. A move relation $(u \cdot s_f, u \cdot s)$ constructed as above is called a **move pair translate**.

Let

$$T_n = \bigcup_{\substack{u \in \mathcal{M}_n \\ (s_f, s) \in MP_n}} (u \cdot s_f, u \cdot s).$$

Thus T_n covers every possible move relation in the game G restricted to heaps of size n or less. By the partioning function, $\varphi(u \cdot s_f), \varphi(u \cdot s) \in \{\mathcal{P}, \mathcal{N}\}$, and we can now check the two correctness requirements listed at the beginning of this section.

Section 9.1.3 of [8] discusses in depth the algorithms used to verify the correctness of the analysis to heap size n.

5.4.3 Misère solver

Aaron Siegel has written an extremely helpful program called *Misère solver* which calculates the indistinguishablility quotient of a misère finite octal game, provided the periodicity of the elements occurs before heap size 10 000. The analysis of 0.3103 in Section 5.4.4 makes us of the program.

The indistinguishably quotient and method from [8] are major breakthroughs in impartial misère game analysis. Coupled with *misère solver*, many previously difficult to analyse misère finite octal games are now, in some sense, solvable, at least within the realm of the game itself.

5.4.4 Using the Indistinguishablility Quotient to Analyse 0.3103

Consider the wild finite octal game 0.3103. Its genus sequence is

$+$	1	2	3	4	5
$0+$	1^{031}	2^{20}	0^{02}	1^{20}	0^{120}
$5+$	1^{031}	2^{1420}	0^{02}	1^{20}	0^{120}
$10+$	1^{031}	2^{1420}	0^{02}	1^{20}	0^{120}
$15+$	$\cdots$				

with $\Gamma(h_0) = 0^{120}$.

Before we begin, we assign e to denote the identity in $\mathcal{F}_{\mathcal{H}_n}$ for each n, which always exists (take h_0). Since h_0 is a next player win, $\varphi(e) \in \mathcal{N}$.

Start with $n = 1$. Take $\mathcal{F}_{\mathcal{H}_1}$. Then we have two elements in our monoid, which we will denote by e and a.

$+$	1
$0+$	a

Since h_1 is the same as a Nim heap of size one and Theorem 2.4.2 gives that $1^{031} + 1^{031} = 0^{120}$, we know exactly how a behaves with other games. Thus we get the relation $a^2 = e$. This relation is not dependent on other elements, and so it will continue throughout the analysis. Since h_1 is a previous player win, $\varphi(a) \in \mathcal{P}$.

Take $n = 3$. Claim the following:

+	1	2	3
0+	a	b	b^2

with the relation $a^2 = e$. Since $\Gamma(h_2)$, $\Gamma(h_3)$ are both tame, and Theorem 2.4.2 tells us how they behave in all sums of games, namely that $2^{20} + 2^{20} = 0^{02}$, we get that $\Phi(h_2) \cdot \Phi(h_2) = \Phi(h_3)$ and $b^3 = b$. Again, this relation is not dependent on other elements, and so it will continue throughout the analysis. Thus, $\mathcal{M}_3$ has presentation $< e, a, b \mid a^2 = e, b^3 = b >$ and can be written as

$$\mathcal{M}_3 \cong \{e, a, b, b^2, ab, ab^2 \mid a^2 = e, b^3 = b\},$$

with outcome partition $\varphi(e), \varphi(b), \varphi(ab) \in \mathcal{N}$, $\varphi(a), \varphi(b^2), \varphi(ab^2) \in \mathcal{P}$.

Continuing as such (and using *misère solver*), we get that

+	1	2	3	4	5
0+	a	b	c^2	c	e
5+	a	d	c^2	c	e
10+	a	d	c^2	c	e
15+	$\cdots$				

with

$$\mathcal{M} \cong < a, b, c, d \mid a^2 = e, b^4 = b^2, b^2c = c, c^3 = ac^2, b^2d = b^3, cd = bc, d^2 = e > .$$

Rewriting, we have that

$$\begin{aligned} \mathcal{M} \cong\ & \{e, a, b, c, d, b^2, b^3, c^2, ab, ac, ad, bc, bd, abd, ab^2, ac^2, bc^2, c^2d, b^2c^2, ab^3 \\ & \mid a^2 = e, b^4 = b^2, b^2c = c, c^3 = ac^2, b^2d = b^3, cd = bc, d^2 = e\}, \end{aligned}$$

with the $\mathcal{P}$ positions a, b^2, c^2, ad, bc.

Now suppose we wish to analyse the game $h_5 + h_7 + h_{12} + h_{18} + h_{19}$. Applying Φ to each of these elements, we see that these are equivalent to the respective monoid elements e, d, d, c^2, and c. Taking their product and applying the relations on $\mathcal{M}$, we get

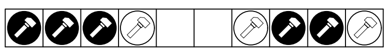
$$\begin{aligned} eddc^2c &= c^3d^2 \\ &= c(cd)^2 \\ &= c(bc)^2 \\ &= (b^2c)c^2 \\ &= c^3 \\ &= ac^2, \end{aligned}$$

which is a Next position. Moreover, the disjunctive sum of heaps of size five, seven, twelve, eighteen, and nineteen behaves like the disjunctive sum of heaps of size one and three within the rules of the game.

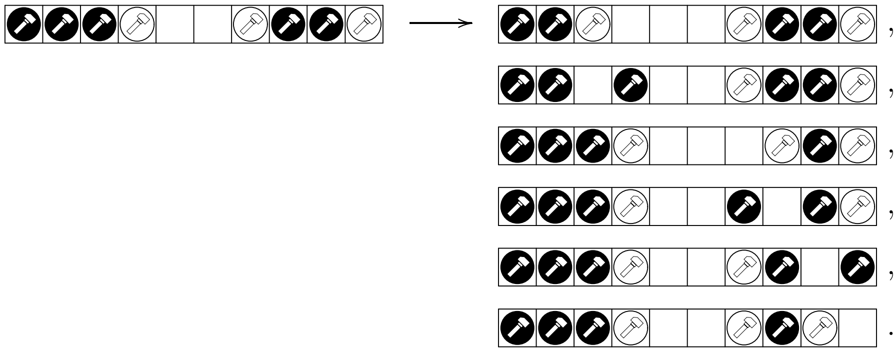
5.5 Indistinguishablility For Other Impartial Games

Ideally, we would like to extend the indistinguishablility quotient method, or some sort of equivalent, to all impartial misère games, not just finite octal games. The difficulty in applying the indistinguishablility method to certain types of games is the notion of when (or indeed even if) the monoid has become periodic. In finite octal games, each position is directly comparable to every other position - thinking of the positions as heaps, either a heap has more tokens or fewer tokens than another heap, and we can order the heaps in a logical manner and check for periodicity in the monoid by pulling the heap into the monoid with the pretending function. However, in some games two positions are not directly comparable, which causes difficulty in determining what it means to become periodic, as the following example shows:

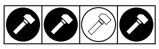
Example 5.5.1. Consider the game of Clobber played on a $1 \times n$ board. Each square contains either a white stone, a black stone, or an empty space. For example:

 .

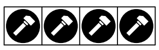
In partizan Clobber, one player may only move white stones while the other player may only move black stones. In impartial clobber on her turn, a player may either move a black or a white stone, and change from turn to turn. A move consists of taking a stone and "clobbering" an adjacent piece of the opposite colour. The clobbered piece is removed from the board, and the clobbering piece takes it place. Pieces cannot move over empty spaces. For the position given above has the following options played impartially:



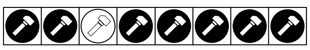
Consider only Clobber positions with no spaces. Any such Clobber position has four immediate successors achieved by appending an  to the beginning, appending a  to the beginning, appending a  to the end, or appending a  to the end. Note that this is different than with finite octals where a heap of size n has only one immediate successor, a heap of size $n + 1$. Moreover, whereas there is only one heap of size n in finite octal games, there are 2^n positions on boards of size $1 \times n$ with no empty spaces, none of which are directly comparable. However, modding out by symmetry, and saying that position A is less than position B if A is a substring of B, we do get a structure, although it is nowhere near as manageable as the structure on heaps in a finite octal game[1]. Not every pair of positions is directly comparable. For example,



and



are not directly comparable, although they do have an least element greater than both of the them, if we mod out by symmetry, namely:

 .

[1]The structure we get for impartial Clobber is a finitely upwards and finitely downwards directed set with binary joins.

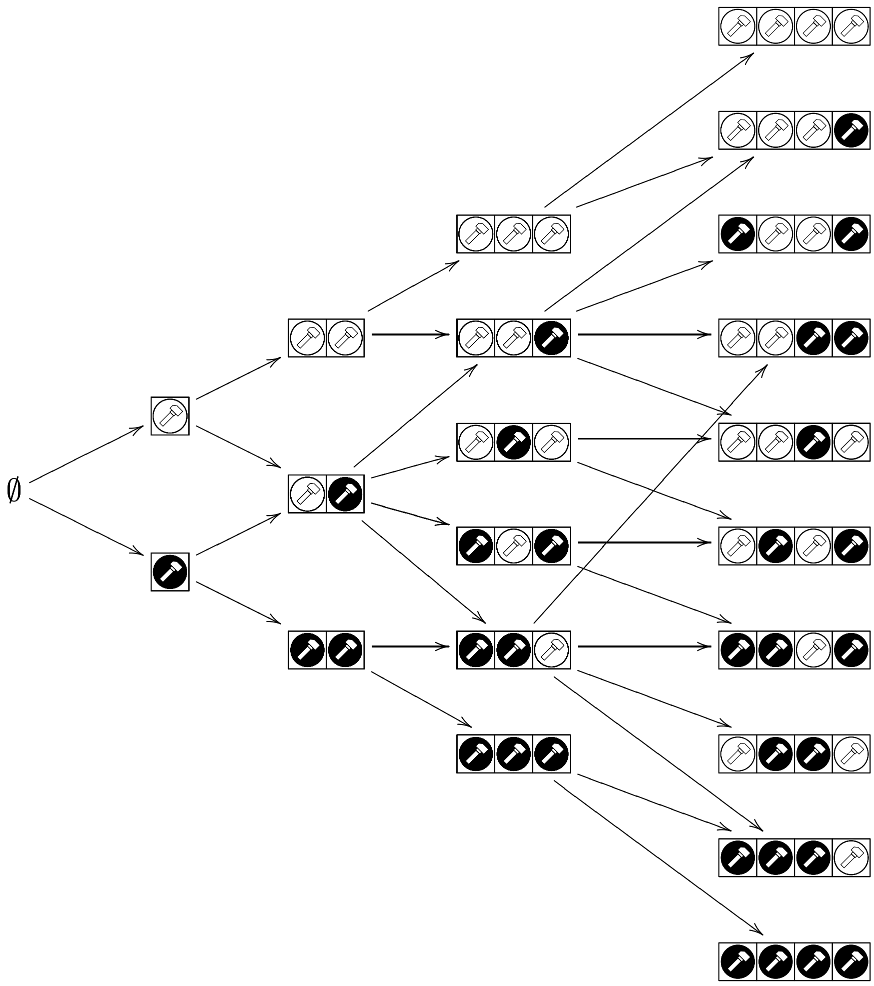
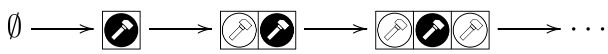


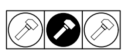
Figure 5.1: The finitely upwards and finitely downwards directed set of impartial Clobber up to 1×4 boards with the $\leq$ relations between adjacent levels drawn in

We can still apply Steps 1) through 7) of Section 5.4 to obtain a monoid. However, what does it mean for this monoid to become periodic?

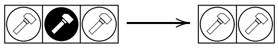
One idea is that, for each path in the directed set, the path becomes periodic. If all paths become periodic, then the monoid is periodic. However, in impartial Clobber, paths are not closed structures under legal moves. A certain position may have a legal move to a position not on the path. For example, consider the path



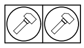
and the position

 .

The following is a legal move:

 ,

but



is not on the path.

Thus we cannot necessarily consider each path in isolation when determining the monoid structure. The paths are interconnected, which might mean that we need not look at every path to determine periodicity.

5.6 Future Work

One goal is to extend the indistinguishablility quotient method to non-octal impartial games, such as the impartial Clobber game given in Example 5.5.1, as well as examine whether indistinguishablility could be extended for use in partizan game analysis.

Another area would be to look at what algebraic (and possibly categorical) results could be applied to the indistinguishablility quotient from monoid and lattice theory. Section 7 of [8] gives some good starting points for further such investigation.

Appendix A

Wild Subtraction Octal Games with Octal Length Six or Less

Given a number $d_1 d_2 \cdots d_n$ in the table below, it represents the octal game $0.d_1 d_2 \cdots d_n$. Those with a $*$ indicate that the subtraction octal game is not domesticatable. Those in **bold** indicate that the subtraction octal game is almost domesticatable. There are no wild subtraction octal games of length two or less.

These results were achieved with the aid of a computer program written by the author which calculates the genera of heaps of finite subtraction octal games.

Table A.1: Wild Subtraction Octal Games with Octal Length Six or Less

Wild Subtraction Octal Games							
123							
0122	0123	1023	1032	1033	1231	1232	1233
1321	1323	1331	2012*	2112*	3101*	3102	3103*
3112	3122	3123	3131	3312			
00122	00123	01022	01023	01032	01033	01122	01123
01221	01222	01223	01231	01232	01233	01302	01312
03012	03022	03023	03112	03122	03123	03201	03202
03211	03212	03301	03302	03311	03312	10023	10032
10033	10122	10123	10132	10133	10202	10203	10212
10213	10231	10232	10233	10321	10322	10323	10331
10332	10333	11032	11033	11203	11212	11213	11221
11223	11231	11233	11303	11331	12022	12023	12032
12033	12102	12112	12123	12133	12202	12203	12212
12213	12301	12311	12321	12322	12323	12331	12332

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Table A.1 – continued from previous page

12333	13032	13033	13102	13103	13112	13123	13211
13212	13213	13231	13232	13233	13303	13331	20101*
20102*	20103*	20121*	20122*	20123*	20301	20311	21101*
21102*	21103*	21121*	21122*	21123*	21301	21311	22012*
22112*	23012*	23112*	30012	30032	30112	30132	31001*
31002	31003*	31011*	**31012**	31013*	**31022**	31023	31031*
31032	31033*	31112	31121	31131	31201*	31202*	31203*
31211	31212	31213	31221*	31222*	31223*	31233	31312
31331	32021	32031	32121	32131	33101*	33103*	33122*
33123	33312						
000122	000123	001022	001023	001032	001033	001122	001123
001221	001222	001223	001231	001232	001233	010022	010023
010032	010033	010122	010123	010132	010133	010203	010213
010221	010222	010223	010231	010232	010233	010321	010322
010323	010331	010332	010333	011032	011033	011122	011123
011203	011213	011221	011222	011223	011231	011232	011233
012022	012023	012122	012123	012201	012211	012221	012222
012223	012231	012232	012233	012301	012311	012321	012322
012323	012331	012332	012333	013021	013022	013023	013121
013122	013123	013201	013202	013211	013212	013301	013302
013311	013312	022001	022011	022101	022111	023001	023011
023101	023111	030102	030112	030121	030221	030222	030223
030231	030232	030233	031102	031112	031121	031221	031222
031223	031231	031232	031233	032012	032021	032112	032121
032201	032202	032211	032212	032301	032302	032311	032312
033012	033021	033112	033121	033201	033202	033211	033212
033301	033302	033311	033312	100023	100032	100033	100122
100123	100133	100212	100213	100231	100232	100233	100312

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Table A.1 – continued from previous page

100313	100321	100322	100323	100331	100332	100333	101022
101023	101032	101033	101122	101123	101132	101133	101221
101222	101223	101231	101232	101233	101321	101322	101323
101331	101332	101333	102021	102022	102023	102031	102032
102033	102121	102122	102123	102131	102132	102133	102202
102203	102212	102213	102301	102302	102303	102311	102312
102313	102321	102322	102323	102331	102332	102333	103022
103023	103032	103033	103121	103122	103131	103132	103201
103211	103221	103222	103223	103231	103232	103233	103301
103311	103321	103322	103323	103331	103332	103333	110032
110122	110123	110132	110133	110203	110212	110213	110221
110223	110231	110233	110312	110313	110321	110322	110323
110331	110332	110333	111032	111033	111203	111213	111231
111331	112031	112032	112033	112121	112131	112132	112133
112211	112212	112213	112231	112232	112233	112302	112303
112311	112312	112313	112331	112332	112333	113021	113023
113031	113033	113121	113131	113201	113203	113211	113212
113213	113221	113223	113231	113233	113303	113311	113331
120221	120222	120223	120231	120232	120233	120321	120322
120323	120331	120332	120333	121002	121003	121021*	121022
121023	121031	121102	121103	121121*	121122	121123	121131
121222	121223	121231	121322	121323	121331	122021	122031
122121	122131	122202	122203	122212	122213	122302	122303
122312	122313	123002	123012	123021	123023	123031	123033
123102	123112	123121	123123	123131	123133	123201	123211
123221	123222	123223	123231	123232	123233	123301	123311
123321	123322	123323	123331	123332	123333	130102	130112
130321	130323	130331	130333	131002	131003	131021	131022

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Table A.1 – continued from previous page

131023	131031	131102	131103	131112	131121	131122	131123
131131	131231	131321	131331	132101	132111	132121	132131
132301	132302	132303	132311	132312	132313	132321	132322
132323	132331	132332	132333	133031	133102	133103	133111
133112	133121	133123	133131	133212	133213	133303	133331
200012*	200112*	201001*	201002*	201003*	201011*	201012*	201013*
201021*	201022*	201023*	201031*	201032*	201033*	201112*	201122
201123	201201*	**201202**	201203*	201211*	**201212**	201213*	201221*
201222*	201223*	201231*	201232*	201233*	201301	201311	203002
203003	203012	203022	203023	203102	203103	203112	203122
203123	203201	203211	203301	203311	210012*	210112*	211001*
211002*	211003*	211011*	211012*	211013*	211021*	211022*	211023*
211031*	211032*	211033*	211112*	211122	211123	211201*	**211202**
211203*	211211*	**211212**	211213*	211221*	211222*	211223*	211231*
211232*	211233*	211301	211311	213002	213003	213012	213022
213023	213102	213103	213112	213122	213123	213201	213211
213301	213311	220101*	220102	220103*	220112	220121*	220122*
220123*	221101*	221102	221103*	221112	221121*	221122*	221123*
222012*	222112*	223012*	223112*	230101*	230102	230103*	230112
230121*	230122*	230123*	231101*	231102	231103*	231112	231121*
231122*	231123*	232012*	232112*	233012*	233112*	300101*	300102
300103*	300121	300122*	300123	300132	300321	300322	300323
301101*	301102	301103*	301121	301122*	301123	301132	301321
301322	301323	302101	302102	302111	302112	302122	302132
303101	303102	303111	303112	303122	303132	310001*	310002
310003*	310011*	310012*	310013*	310022*	310023	310031*	310032*
310033*	310101*	**310102**	310103*	310111*	**310112**	310113*	**310121**
310122*	**310123**	310131*	**310132**	310133*	**310201**	**310202**	**310203**

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Table A.1 – continued from previous page

310211	310212	310213	310221*	310222*	**310223**	310232	310233
310301	**310302**	**310303**	310311*	310312*	310313*	310321*	**310322**
310323	310331*	310332*	310333*	311002	311012	311021	311023
311031	311033	311101*	311102	311103*	311112	311121	311122*
311123	311131	311202	311212	311221	311231	311312	311331
312001*	312002*	312003*	312011*	312012*	312013*	312021*	312022*
312023*	312031*	312032*	312033*	312101*	312102	**312103**	312111
312112*	312113	312122*	312123	312131	312132*	312133	312201*
312202*	312203*	312211*	312212*	312213*	312221*	312222*	312223*
312231*	312232*	312233*	312301*	312302*	312303*	312311*	312313*
312322	312323	312332*	312333	313003	313013	313023	313033
313102*	313103	313111	313112	313121	313122*	313123	313131
313312	313331	320211	320311	321211	321311	322023	322033
322123	322133	323023	323033	323123	323133	330021	330031
330121	330131	331002	331003	331011*	**331012**	331013*	331022
331023	331031*	331032*	331033*	331102	331103	331122	331123
331201*	331202*	331203*	331211*	331212	331213*	331221*	331222*
331223*	331232*	331233	331311	331312	331331	333101*	333103*
333122*	333123	333312					

Appendix B

Genera of 2 x m Toppling Tower positions with eight or fewer towers

We present here a chart of genera of 2 $\times m$ Toppling Tower positions with eight or fewer towers. Because of Theorem 2.4.2, Proposition 2.4.3, and Proposition 2.4.4, we need only show the genera of eight or fewer contiguous towers or the disjunctive sum of five contiguous towers and three contiguous towers, as the genera of the rest can be calculated by the appropriate use of the above Theorem and Propositions. Genera with a * indicate that the position is tame (genus is a tame value **AND** all options are tame).

These results were achieved by hand by the author.

Table B.1: Genera of Toppling Towers in $2 \times m$ boards with eight towers or less

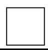
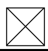
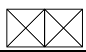
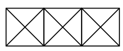
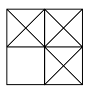
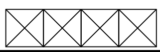
n	Towers	Genera
0		0^{120} *
1		1^{031} *
2		2^{20} *
3		3^{31} *
		3^{31} *
4		4^{46} *

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Table B.1 – continued from previous page

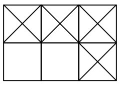
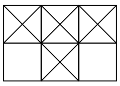
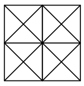
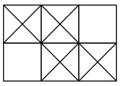
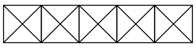
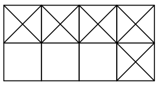
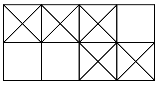
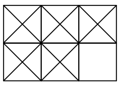
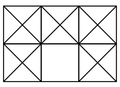
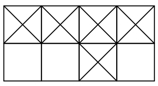
n	Towers	Genera
		4^{46} *
		2^{20} *
		0^{02} *
		1^{031} *
5		5^{57} *
		5^{57}*
		5^{146}
		5^{146}
		1^{531}
		5^{57}*

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Table B.1 – continued from previous page

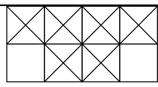
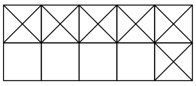
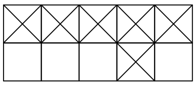
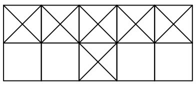
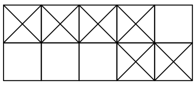
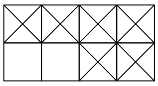
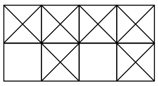
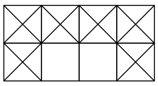
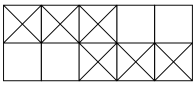
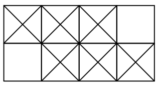
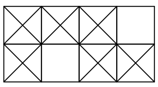
n	Towers	Genera
6		6^{657}
		$6^{64}*$
		$4^{46}*$
		$2^{20}*$
		6^{657}
		6^{657}
		4^{46}
		6^{046}
		6^{657}
		0^{02}
		6^{657}

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Table B.1 – continued from previous page

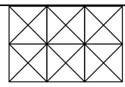
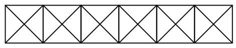
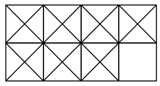
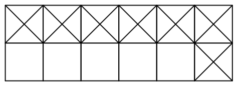
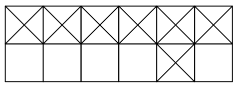
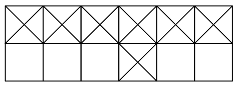
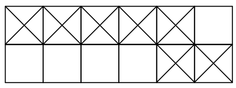
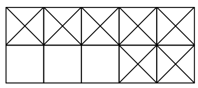
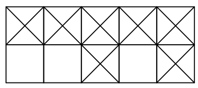
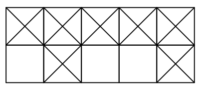
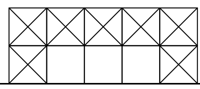
n	Towers	Genera
		2^{1520}
		$6^{64}*$
7		$7^{78(10)}$
		$7^{75}*$
		$7^{75}*$
		$7^{75}*$
		$7^{78(10)}$
		$7^{78(10)}$
		7^{75}
		7^{75}
		1^{731}

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Table B.1 – continued from previous page

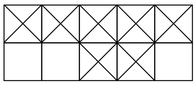
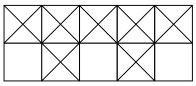
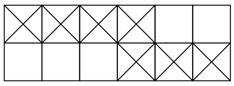
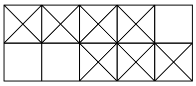
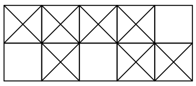
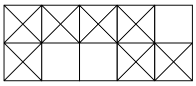
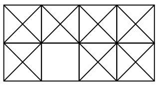
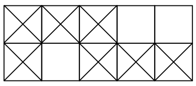
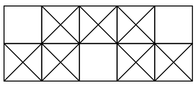
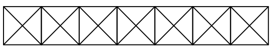
n	Towers	Genera
		$7^{78(10)}$
		5^{57}
		$7^{78(10)}$
		$7^{78(10)}$
		7^{746}
		$7^{78(10)}$
		$7^{78(10)}$
		3^{31}
		2^{20}
		$7^{75}*$

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Table B.1 – continued from previous page

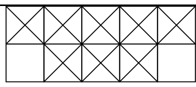
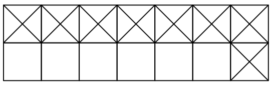
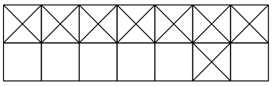
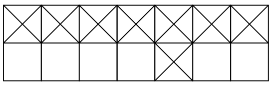
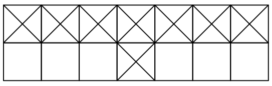
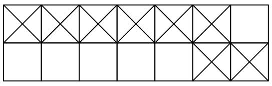
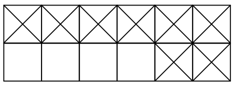
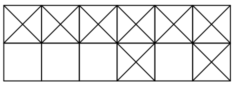
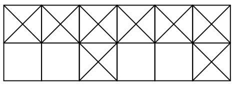
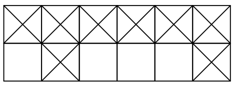
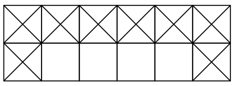
n	Towers	Genera
8		1^{531}
		$8^{8(10)}*$
		$6^{64}*$
		$8^{8(10)}*$
		$6^{64}*$
		8^{57}
		4^{57}
		6^{64}
		$8^{8(10)}$
		6^{64}
		$1^{031}*$

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Table B.1 – continued from previous page

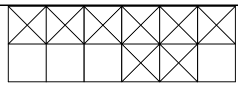
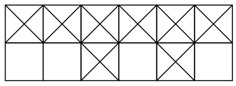
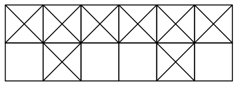
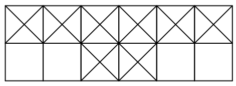
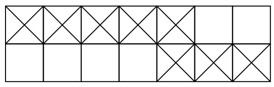
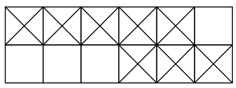
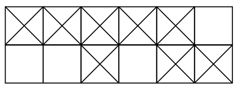
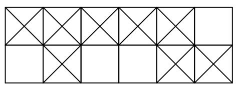
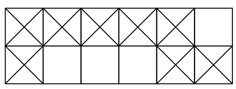
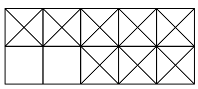
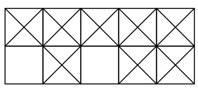
n	Towers	Genera
		8^{57}
		6^{64}
		6^{64}
		1^{531}
		$8^{8(10)}$
		6^{657}
		5^{57}
		$6^{69(11)}$
		8^{875}
		0^{820}
		$6^{69(11)}$

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Table B.1 – continued from previous page

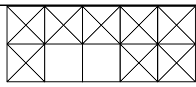
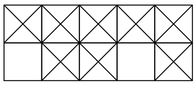
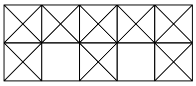
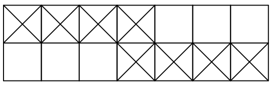
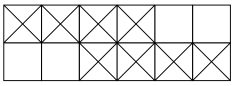
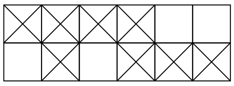
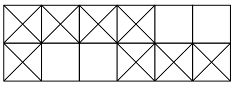
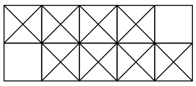
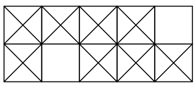
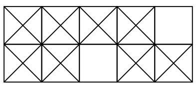
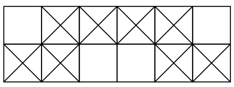
n	Towers	Genera
		5^{57}
		5^{57}
		6^{146}
		1^{13}
		0^{02}
		$6^{69(11)}$
		3^{31}
		1^{13}
		6^{64}
		5^{57}
		5^{57}

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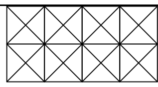
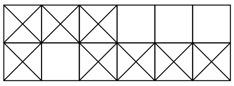
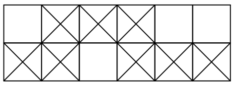
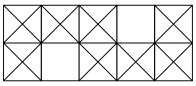
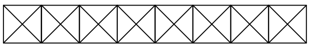
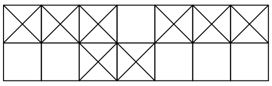
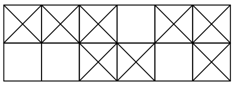
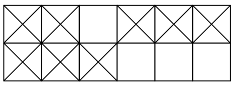
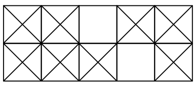
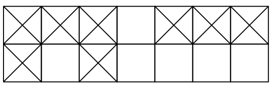
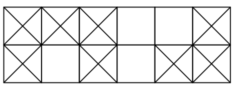
Table B.1 – continued from previous page

n	Towers	Genera
		0^{02}
		2^{1920}
		8^{57}
		1^{13}
		$8^{8(10)}$
		6^{57}
		6^{57}
		6^{57}
		6^{57}
		2^{20}
		2^{20}

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List of Symbols Used

$0.a_1a_2a_3\overline{a_4a_5a_6}$	$0.\ \ a_1a_2a_3\ \ a_4a_5a_6\ \ a_4a_5a_6\ \ a_4a_5a_6\ \ \cdots$	10
$\overset{M}{\equiv}$	$G \overset{M}{\equiv} H$ if $\Gamma(G) = \Gamma(H)$	58
$\mathcal{F}_{\mathcal{H}}$	the free Abelian monoid on $\mathcal{H}$	101
$\mathcal{F}$	The Free Functor	101
$\mathcal{G}^L$	the set of Left options of a position G	2
$\mathcal{G}^R$	the set of Right options of a position G	2
$\mathcal{G}^+(G)$	0 if G has no options $\text{mex}\{\mathcal{G}^+(G') \mid G'$ is an option of $G\}$ else	16
$\mathcal{G}^-(G)$	1 if G has no options $\text{mex}\{\mathcal{G}^-(G') \mid G'$ is an option of $G\}$ else	16
$\Gamma(G)$	the genus of G	19
$\mathcal{H}$	$\{h_1, h_2, h_3, \cdots\}$	101
$\mathcal{H}_n$	$\{h_1, h_2, h_3, \cdots, h_n\}$	101
$\mathcal{L}$	Left has a winning strategy regardless of moving first or second	4
$\mathbf{mex}\{\mathcal{X}\}$	the least ordinal not in $\mathcal{X}$	11
MP_n	move pairs to heap size n	107
$\mathcal{N}$	the Next player to move has a winning strategy	4
$\mathbb{n}$	a Nim heap with n tokens	2

$\oplus$	Nim sum	12
$\mathcal{N}(G)$	For $G = 0.d_1 d_2 \cdots d_n$, a finite subtraction octal game, $\mathcal{N}(G) = 0.d_1 d_2 \cdots d_n \bar{3}$	75
$\mathcal{P}$	the Previous player to move has a winning strategy	4
$\mathcal{Q}(G)$	$\mathcal{F}_{\mathcal{H}}/\rho$	103
$\mathcal{R}$	Right has a winning strategy regardless of moving first or second	4
$u\ \rho\ v$	u is indistinguishable from v	102
$*$	the game $\{0\|0\}$	8
$*_n$	$\{0, *, *_2, *_3, \cdots, *_{n-1} \| 0, *, *_2, *_3, *_{n-1}\}$	11
$\mathcal{T}$	the set of tame games	59
$[\mathcal{T}]$	$\mathcal{T}/\overset{M}{\equiv}$	59

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