When You Want To Lose:
A Semigroup Approach to Misere Games

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I. Background

Combinatorial game theory is an area of mathematics that is full of opportunities to research for any level of education. The nature of the field allows one to add a variation to a game which may make it more or less interesting to study. Typically the winner of a game is the player to make the last move. In the misère version of games, the player that moves last loses. Such a simple variation may seem like it would not affect the strategy of play very much; however, for most games that is not the case.

Nim is a game that starts with heaps of stones, and players take turns removing any number of stones they would like from one heap. In normal play the winner is the player to remove the last stone. The strategy for this game is to leave the nim-sum of the heaps as zero after your turn. (The nim-sum is binary addition of the number of stones in each heap without carrying.) This will guarantee that for any move your opponent makes, you will also have a move. The interesting thing about nim is that it has been proven that when using normal play, any impartial combinatorial game relates to nim. [2]

The strategy for misère play in nim is the same as normal play, except when a player's move is going to leave only heaps of zero and one stone. The reasoning behind this slight change of strategy is simple. In both normal and misère nim, all single heaps of two stones or greater are a win for the next player to move (an N-position). For normal nim a heap of one stone is a N-position and a heap of zero is a P-position (a win for the player that moved previously); these two outcomes switch in misère nim.

This change in the outcomes affects the operations on Gundy values as well. The Grundy value in normal play is a value given to impartial games based on the options of the player. In misère play, relatively few games can be given a Grundy value because the game's value is G=*m where m is the minimum excluded value (mex) of the options the player has from game G with the added stipulation that at least one of G's options is a *0 or a *1. When dealing with subtraction games, this does not cause any complications [3, p 442], but this causes chaos in many other games. Another change in the Grundy values is that the nim-sum of two values is also only defined for when *0 or *1 is added to another nimber. Otherwise, addition is undefined. For example *2 + *2 in normal play is equal to *0, but in misère *2 + *2 cannot be simplified further. These new versions of the addition and mex rules mean that depending solely on the Grundy values to find the outcomes of games most likely won't be sufficient. In other words, not all impartial combinatorial games relate to nim when following misère play. [5]

Recently, Thane Plambeck made a huge leap in the progress of misère games by looking at games in terms of equivalence classes based on possible game positions instead of all game positions. The traditional definition of games being equivalent is that game G equals game H when G+X = H+X for all games X. According to Plambeck's approach, if one is trying to analyze a game in a specific context, then one does not need a definition of equivalence to be this strong. Instead, a set A is defined where A is a collection of games closed under addition. Additionally for any game in A, any option of a game G in A must also be in A. Now a new kind of equivalence $=_A$ can be defined as follows: if G and H are games in the set A, then G $=_A$ H if and only if G+X $=_A$ H+X for all games X in A. [5]

This new definition of equality simplifies the process of determining the value of a game G because it reduces the number of games that have to be added to it in order to compare outcomes. Using an example from Aaron Siegel's paper [5], let A be the set of all sums of * and *2. There is a total of six equivalence classes formed by this A (0, *, *2, *2+*, *2+*2, *2+*2+*). It can be shown that any other sum is equivalent to one of those classes based on the new definition of equivalence. This set of equivalence classes is referred to as the misère quotient (Q(A)). It also can be shown that any Q(A) is a commutative monoid. Every game must include 0 (the identity element), and the operation is clearly commutative (*2+* = *+*2). The set A is closed by definition, and it is also clearly associative. Since A is closed under addition and includes any option of G in A, we can also say that for this example Q(A)=Q(*2).

The collection of equivalence classes is commonly referred to through an isomorphism. Begin by defining the isomorphism Φ by Φ(0) = 1, Φ(*) = a and Φ(*2) = b. Normally * + *2 = *3, but *3 is not included in the set A. Φ(* + *2) = ab. In misère *2 + *2 is already in its simplified form, and Φ(*2 + *2) = b^2. Hence Q(A) = [1, a, b, ab, b^2, ab^2] where a^2=1 and b^3=b. Note also that in this example, there are two classes that are P-positions: a and b^2. The convenience of this information is if one was looking at a game that was determined by a finite number of *'s and *2's, the value of the game through the isomorphism is $a^n b^m$ where n is the number of *'s and m is the number of *2's. Then use the identities for a and b to quickly determine the outcome of the game.

At this point a game can be classified as one of two types of games: wild or tame. A game is tame if its misère quotient is isomorphic to a misère quotient from the game of nim. The quotients for nim are denoted by T_n where $T_n = Q(*2^{n-1})$. Hence, 2^{n-1} stones is the maximum number of stones in a nim heap when playing this game. Returning to our previous example with Q(*2), Q(*2) is tame and is isomorphic to T_2. T_n equals $< a, b_1, b_2, \ldots, b_{n-1} \mid a^2 = 1, b_i^3 = b_i, b_1^2 = b_2^2 = \ldots = b_{n-1}^2 >$ where a and b_1^2 are P-positions. Thus to find $\Phi(*m)$, write m in binary $(\ldots c_2c_1c_0)$ then $\Phi(*m) = a^{c0}b_1^{c1}b_2^{c2}\ldots$ Note that when dealing with small scales b_1 is often written as just b, b_2 is replaced by c, b_3 is replaced by d, etc. If a misère quotient is not isomorphic to a misère quotient for nim, then it is classified as wild. $R_8 = <a, b, t \mid a^2 = 1, b^3 = b, t^2 = b^2, bt = b>$ with P-positions a and b^2 is a special example of a wild game because it is the smallest wild misère quotient. The only tame quotients smaller than R_8 are T_0, T_1, and T_2. [5]

Thane Plambeck has focused much of his research at this point on octal games [5]. He has been using a periodicity theorem for octal games that has been expanded for misère play. Our goal is to apply the idea of misère quotients to games that are not necessarily octal games, including possibly partizan games.

II. EndNim

The main game examined for this work is EndNim. As described by *Lessons in Play* [1], a game of EndNim consists of a row of stacks of boxes. On a player's turn, he/she may remove any number of boxes from one of the two stacks on the ends of the row, up to the entire stack. In misère play, the losing player is the player to remove the last box.

Upon initial examination of the game, determining the values of rows was relatively easy. For each arrangement of boxes in stacks up to three boxes per row, the value of each game was simply determined by computing the mex of the options. Let E equal the set of possible games in EndNim. Then define the isomorphism Φ such that $\Phi(*0) =_E 1$, $\Phi(*1) =_E a$, and $\Phi(*2) =_E b$. For simplicity of notation, let [x,y,..z] represent a row of boxes with x boxes in the first stack, y boxes in the second stack, … and z boxes in the last stack. Recall that a player is limited to removing boxes from either the first or the last stack. This simplifies the analysis of a game. It also means that a row is equal to its mirror image (e.g. [x,y,z] = [z,y,x]). Since the options of all the possible rows of up to three boxes always include a 1 or 0, their values can be computed through the mex of the options; therefore this game is clearly isomorphic to T_2.

Row	**Φ Value**
[0]	1
[1]	a
[2]	b
[1,1]	1
[3]	ab
[2,1]	ab
[1,1,1]	a

Table 1: The values of all rows containing up to three boxes.

When increasing the possibilities to a maximum of four boxes, the value of *4 or $\Phi(*4) =_E c$ is introduced. This means that if EndNim with four boxes is tame, it is isomorphic to T_3. There are also two games that cannot be computed through the mex due to the lack of *1 and *0 in the list of options. Those games are [2,2] and [1,2,1]. Notice that in [2,2] both stacks can be accessed by the players. In fact, it can be treated as [2] + [2], or $b \bullet b = b^2$. Unfortunately, this trick does not work for [1,2,1] because players are not able to access the middle stack of two boxes until after one of the stacks on the end have been removed. While continuing to examine EndNim through a maximum of seven boxes per row, there are many more of these cases that cannot be quickly determined though the mex. In order to determine the values of an unknown game G (and if EndNim remains tame) we must observe the outcomes of G added to the other possible games of EndNim. If they correspond to the outcomes of a tame value H added to the other possible games, then $G =_E H$. If no game H exists, then G must be a wild component of EndNim.

In order to make these computations easier, first create the operation table for T_3 (see Table 2).

(•)	1	a	b	b^2	ab	ab^2	c	ac	bc	abc
1	1	a	b	b^2	ab	ab^2	c	ac	bc	abc
a	a	1	ab	ab^2	b	b^2	ac	c	abc	bc
b	b	ab	b^2	b	ab^2	ab	bc	abc	c	ac
b^2	b^2	ab^2	b	b^2	ab	ab^2	c	ac	bc	abc
ab	ab	b	ab^2	ab	b^2	b	abc	bc	abc	c
ab^2	ab^2	b^2	ab	ab^2	b	b^2	ac	c	abc	bc
c	c	ac	bc	c	abc	ac	b^2	a	b	ab
ac	ac	c	abc	ac	bc	c	a	b^2	ab	b
bc	bc	abc	c	bc	abc	abc	b	ab	b^2	ab^2
abc	abc	bc	ac	abc	c	bc	ab	b	ab^2	b^2

Table 2: Operation Table for T_3

Recall that in T_3, $a^2 = 1$, $b^3 = b$, $b^2 = c^2$, and the P-outcomes are a and b^2 [5]. Notice that 1 is the identity, but b^2 also acts like an identity for all values except 1 and a. Replace the sums in the table with their outcomes to obtain the desired information.

(•)	1	a	b	b^2	ab	ab^2	c	ac	bc	abc
1	N	P	N	P	N	N	N	N	N	N
a	P	N	N	N	N	P	N	N	N	N
b	N	N	P	N	N	N	N	N	N	N
b^2	P	N	N	P	N	N	N	N	N	N
ab	N	N	N	N	P	N	N	N	N	N
ab^2	N	P	N	N	N	P	N	N	N	N
c	N	N	N	N	N	N	P	N	N	N
ac	N	N	N	N	N	N	N	P	N	N
bc	N	N	N	N	N	N	N	N	P	N
abc	N	N	N	N	N	N	N	N	N	P

Table 3: Outcome Operation Table for T_3

Now we determine the outcomes of [1,2,1] when added to the other possible games in T_3 (Table 4). Observe that the outcomes for [1,2,1] are the same as b^2, so $[1,2,1] =_E b^2$ where E is the set of all games that occur in EndNim (up to seven boxes per row).

(•)	1	a	b	b^2	ab	ab^2	c	ac	bc	abc
Φ ([1,2,1])	P	N	N	P	N	N	N	N	N	N

Table 4: The outcomes when [1,2,1] is added to all possible games in T_3.

While evaluating individual positions in EndNim, there were many tricks discovered that made the process of evaluating positions occur more quickly and easily. A few are listed and proven below as part of an attempt to prove EndNim is always tame.

III. Is EndNim Tame?

Lemma 1: At most one of b^2 and ab^2 results in a P-outcome when added to game G.

Proof: Suppose that $G \bullet b^2$ was a P-position. $G \bullet b^2$ is also an option of $G \bullet ab^2$, hence $G \bullet ab^2$ has a P-option which makes $G \bullet ab^2$ an N-position.

Now suppose that $G \bullet ab^2$ is a P-position. That means all options of $G \bullet ab^2$ are N positions, which includes $G \bullet b^2$. □

Lemma 2: At most one of a, b_1, ab_1, b_2, ab_2, b_1b_2, ab_1b_2… results in a P-outcome when added to game G.

Proof: All of those values can be represented by a single stack of stones. Let H represent the first (smallest) stack such that G + H is a P-outcome. For each J > H, G + H is an option of G + J. Thus G + J has a P-option and must be an N-position. □

Theorem 1: When determining the outcomes of game G added to the previously determined games (a, b, b^2, etc) at most two of the outcomes are P and the rest are N.

Proof: Lemma 1 and Lemma 2 are an exhaustive list of the options that can be added to G, so at most one from Lemma 1 and at most one from Lemma 2 implies that at most two of the outcomes will be P. □

Theorem 2 is not a necessary element in the attempted proof of EndNim being tame. Instead it was an interesting result that merits a slight deviation from the ultimate goal.

Theorem 2: A game of [1,n,1], where $n > 1$, is equivalent to b^2.

Proof: When faced with the game [1,n,1], the first player (let's call her Alice) only has one option: remove a stack with one box. The second player (let's call him Bob) can respond and win by removing the entire stack of n boxes, which forces Alice to take the last box. Thus $[1,n,1] + \Phi^{-1}(1) =_E [1,n,1]$ is P.

Now examine the game of $[1,n,1] + \Phi^{-1}(b^2)$. Since any b^2 game has the same outcome when added to another game, we can use [2,2] to represent b^2. Hence when faced with [1,n,1] + [2,2] Alice has three options for removing boxes.

a. [n,1] + [2,2]

Bob could win by removing n-1 boxes from the first stack in the first row. This leaves $a^2 \bullet b^2 = 1 \bullet b^2 = b^2$, which is a P-position.

b. [1,n,1] + [2]

We have already determined that [1,n,1] is a P-position, so Bob would respond by removing the single stack of two boxes.

c. [1,n,1] + [1,2]

Bob can win by removing one box from the second stack in the second row. This leaves $[1,n,1] + [1,1] =_E [1,n,1] + \Phi^{-1}(a^2) =_E [1,n,1] + \Phi^{-1}(1) =_E [1,n,1]$.

Since all of Alice's opening options result in Bob winning the game, $[1,n,1] + \Phi^{-1}(b^2)$ is also a P-position. By Lemmas 1 and 2, the remainding games in T_3 added to [1,n,1] must result in an N-outcome. Therefore the outcomes of [1,n,1] match the outcomes of b^2 and $\Phi([1,n,1]) =_E b^2$. □

Notice that Lemma 1 and Lemma 2 eliminate a lot of possible combinations of outcomes from the collection of EndNim games, which leads to the question: Is it possible to eliminate the remaining extraneous combinations of outcomes and declare EndNim tame for any finite number of boxes? In order for this to happen, we need the following lemma and the proof of the following conjecture.

Lemma 3: For any game G, there exists at least one game H such that G + H is a P-position.

Proof: For any game G, G • G is a P-position with the exception of G = a or G = 1. In order to show this, simply refer to the method of winning for misère nim. The second player simply matches the first player's moves until there are only stacks of zero and one boxes remaining. For the cases of G = a or G = 1, refer to Table 3.□

Notice that Lemma 3 does not help define what G is equivalent to, it only guarantees that there is no game that always results in a N-outcome when added to another game.

Conjecture: One of 1 • G = G and a • G is a P position if and only if one of b^2 • G and ab^2 • G is a P-position.

The proof of this proposition would be sufficient to prove that EndNim is tame. The proof in combination with previous theorems would also eliminate the need for Theorem 3. Unfortunately such a proof has not been completed because here another difficulty when using misère play instead of normal play arises. In normal play, it is usually sufficient to know the outcomes of two games in order to determine the outcome of their sum (see Table 5). In misère play, there are no such guarantees. A quick way to verify this is to refer to Table 3.

a.

+	P	N
P	P	N
N	N	?

b.

+	P	N
P	?	?
N	?	?

Table 5: When adding games with known outcomes in normal play (a.), some of the resulting outcomes are known, but in misère play (b.), none of these combinations are certain.

Theorem 3: If G • ab_1^2 or G • b_1^2 is P, then G • b_1, ab_1, b_2, ab_2, b_1b_2, ab_1b_2… are N-positions.

Proof: Recall that $\Phi(*m) = a^{c0}b_1^{c1}b_2^{c2}$ where $\ldots c_2c_1c_0$ is m in binary. Then for tame game $a^{c0}b_1^{c1}b_2^{c2}$ can be represented by a single stack of m boxes. Also note that when dealing with a game that consists of only two stacks of boxes, the players are able to move with both stacks. Therefore the value of the game is the sum of the values of the two stacks.

Assume that G • ab_1^2 is P. Observe that ab_1^2 can be an option of b_2, ab_2, b_1b_2, ab_1b_2… (*4, *5, *6, *7…) by writing those games as the sum of two stacks. Let H be a game from b_2, ab_2, b_1b_2, ab_1b_2… Since none of these have a squared term ($b_n^2 = b_1^2$), we know that H can be a single stack. An equivalent game J can be created that consists of exactly two stacks: stack A and stack B. Let stack A equal ab1 and stack B equal game H • a. If H contains a b1 term, remove it from stack B, otherwise add it to stack B. This guarantees that stack B has at least four boxes and games J and H are equivalent. The way J was designed, $ab_1 \bullet b_1 = ab_1^2$ is an option of J. Since G • ab_1^2 is P and an option of G • J, then G • J is N. Note that $\Phi([3, 2] + [2]) =_E ab_1^2 \bullet b_1 = ab_1$ can be turned into ab_1^2 by removing a stack of two boxes and $\Phi([2, 2] + [2]) =_E b_1 2$ •

$b_1 = b_1$ can be turned into ab_1^2 by removing a single box. Hence G • b_1 and G • ab_1 are also N-positions

Now assume that G • b_1^2 is P. A similar argument follows except begin with stack A equal to b_1 and stack B equal to H. In the special cases remove the stack of three boxes from [3, 2] + [2] and remove a stack of two boxes from [2, 2] + [2]. □

This leaves four special cases as listed below in Table 6. These cases are the only remaining possibilities for wild games to occur in EndNim. If it can be shown that these positions cannot occur in EndNim, then EndNim is tame for any number of boxes.

(•)	1	a	b_1	b_1^2	ab_1	ab_1^2	b_2	ab_2	b_1b_2	ab_1b_2	…
G_1	P	N	N	N	N	N	N	N	N	N	…
G_2	N	P	N	N	N	N	N	N	N	N	…
G_3	N	N	N	P	N	N	N	N	N	N	…
G_4	N	N	N	N	N	P	N	N	N	N	…

Table 6: The remaining outcome combinations that may or may not occur in EndNim.

IV. Other Games

The misère variations of other games were also investigated including both the partizan and impartial versions of maze and maize, and toppling dominoes. Toppling dominoes is a game that is set up with a row of dominoes that are traditionally colored black or white to represent which player may use them, as well as gray to represent both players being able to use them. On a players turn, he/she selects a domino and knocks it to the left or right which also knocks over any of the other dominoes to the left or right. When playing with more than one game, the dominoes from one game do not affect the dominoes in the other games [1]. For impartial toppling dominoes (all dominoes are grey), the game is identical to nim. The player can choose to remove a specific number of dominoes from a row just like he/she could choose to remove a specific number of stones from a nim heap. Therefore impartial toppling dominoes is tame. Similarly, the partizan version of toppling dominoes would be identical to partizan nim where the stones are referred to as stacks with a specific order instead of generic heaps.

Maze is a game that is played on an angled grid. The edges of the grid are walls, and other walls may be added within the grid. The game is played with a marker somewhere on the grid. One player can only move the marker down towards the left, and the other can only move it down towards the right (unless an impartial version is played, then both players may move in either direction). The marker can be moved any number of spaces in one turn, as long as they are in the same direction. The game ends when the marker cannot move because it is blocked by a wall. Maize is a variation that restricts movement to one space per turn. When ma(i)ze games are restricted to plain rectangular boards, simple patterns arise in the game values for both the partizan and impartial versions; however the more interesting ma(i)ze games are on non-rectangular boards and/or boards with walls. These were more difficult to investigate

V. Conclusion

The misère version of several non-octal games has been investigated, with this focus on the game of EndNim. Through an exhaustive list, the game of EndNim has been proven tame for all games containing a maximum of seven boxes. Further examination of EndNim eliminated all but four possible games (based on outcomes when summed with other games) that could prevent EndNim from being entirely tame. More research is necessary to determine whether or not those games exist in EndNim. If they do exist, then it would also be worth determining exactly what the equivalence class(es) of the corresponding games look like.

The expansion of the idea of the misère quotient for partizan games should also be further investigated. A partizan misère nim would have to be clearly defined and studied to determine if the terms tame and wild could still apply to partizan games. It is also important to note that just as the mex rules change when dealing with impartial misère games, the simplification rules change when dealing with partizan misère games.

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Appendix: EndNim Values for Games with up to Seven Boxes

Each game G is followed by Φ(G).

0 boxes	
[0]	1
1 box	
[1]	a
2 boxes	
[2]	b
[1,1]	1
3 boxes	
[3]	ab
[2,1]	ab
[1,1,1]	a
4 boxes	
[4]	c
[3,1]	b
[2,2]	b^2
[2,1,1]	b
[1,2,1]	b^2
[1,1,1,1]	1
5 boxes	
[5]	ac
[4,1]	ac
[3,2]	ab^2
[3,1,1]	1
[1,3,1]	b^2
[2,2,1]	ab^2
[2,1,2]	b^2
[2,1,1,1]	ab
[1,2,1,1]	ab^2
[1,1,1,1,1]	a
6 boxes	
[6]	bc
[5,1]	c
[4,2]	bc
[4,1,1]	ab
[1,4,1]	b^2
[3,3]	b^2
[3,2,1]	b
[3,1,2]	ab^2
[1,3,2]	b
[3,1,1,1]	b
[1,3,1,1]	b
[2,2,2]	b
[2,2,1,1]	b^2
[2,1,2,1]	b
[2,1,1,2]	b^2
[1,2,2,1]	b^2
[2,1,1,1,1]	b
[1,2,1,1,1]	b^2
[1,1,2,1,1]	b^2
[1,1,1,1,1,1]	1
7 boxes	
[7]	abc
[6,1]	abc
[5,2]	abc
[5,1,1]	ac
[1,5,1]	b^2
[4,3]	abc
[4,2,1]	c
[4,1,2]	c
[1,4,2]	ab^2
[4,1,1,1]	c
[1,4,1,1]	ab^2
[3,3,1]	ab^2
[3,1,3]	b^2
[3,2,2]	ab
[2,3,2]	b^2
[3,2,1,1]	ab
[3,1,2,1]	ab
[3,1,1,2]	ab^2
[2,3,1,1]	a
[1,3,2,1]	b^2
[1,3,1,2]	ab
[2,2,2,1]	ab
[2,2,1,2]	ab
[2,2,1,1,1]	ab^2
[2,1,2,1,1]	ab
[2,1,1,2,1]	b
[2,1,1,1,2]	b^2
[1,2,1,2,1]	b^2
[1,2,2,1,1]	ab^2
[2,1,1,1,1,1]	ab
[1,2,1,1,1,1]	ab^2
[1,1,2,1,1,1]	ab^2
[1,1,1,1,1,1,1]	a